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Finite Equational Bases for Finite Algebras in a Congruence-Distributive Equational Class*

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Does every finite algebraic system A with finitely many operations possess a finite list of polynomial identities (laws), valid in A, from which all other such identities follow? Surprisingly, no (R. C. Lyndon, 1954). The answer is, however, affirmative for various particular kinds of algebraic systems, such as finite groups (Oates and Powell), finite lattices, and even finite lattice-ordered algebraic systems (McKenzie). The purpose of the present paper is to provide a sufficient condition that guarantees an affirmative answer without referring to any particular kind of operation: It is sufficient for A to be a finite member of an equational class of algebraic systems whose congruence lattices are distributive. The proof is constructive. Applications include the case of lattice-ordered algebraic systems.

1. INTRODUCTION

1.1. *Finite Equational Bases*

For a given algebra (algebraic system) A, one common object of study is the set of polynomial identities of A (laws of A, identical relations of A)—the polynomial equations that hold for all elements of A. In this regard, a natural question arises: (Q) Does A have a "finite equational basis," i.e., a finite list of (polynomial) identities of A that imply all the other identities of A?

For example, if A is a two-element group, regarded as an algebra with operations of multiplication, inversion, and a constant operation e, then A does have a finite equational basis, consisting of the identity $x^2 = e$, together with the several laws that serve to define the class of all groups. From these, all other identities of A, such as the commutative

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law and, say, $(xy)x = y$, can be derived. (In identities, universal quantification is to be understood.)

For *finite* algebras A, an intuitively plausible conjecture would be that the answer to (Q) should always be "yes." After all, should not all properties of a finite algebra be finitely describable? Surprisingly, however, this conjecture is false: In 1954, Lyndon [49] constructed a seven-element algebra, with one binary operation, for which there is *no* finite equational basis. Subsequently, Visin [81] found a four-element example and Murskiĭ [64] found a three-element example. Perkins [67] showed that the answer to (Q) is negative even for an easily describable six-element semigroup: the semigroup of 2×2 matrices comprised by the identity matrix, the zero-matrix, and the four matrices whose entries are 0, 0, 0, 1 in some arrangement.

In a sense, the supposedly plausible conjecture founders on two "infinite" aspects of the problem: (1) The identities of A involve arbitrarily large numbers of variables, and (2) the "implication" of all identities of A by a proposed finite list means that all algebraic systems B, finite or infinite, that satisfy the list must then satisfy all identities of A. (Equivalently, the finite list must yield all identities of A by formal derivations making no reference to A at all.)

Nevertheless, a finite equation basis *is* known to exist in the following nontrivial cases: All algebras with two elements and finitely many operations [48]; all finite groups [65]; finite Heyting algebras (de Jongh, reported in [80]); three-element semigroups and commutative semigroups [67]; finite lattices and even finite lattices with finitely may additional operations [57]; finite rings [45]; finite commutative Moufang loops [20]; and finite loops with no proper nontrivial subloops [51]. For further comment, see the informative surveys of Tarski [77] and MacDonald [50].

1.2. *Congruence Distributivity and the Main Theorem*

In the meantime, B. Jónsson had discovered a pair of surprising theorems about equational classes (varieties, primitive classes, classes defined by identities) of algebras whose lattices of congruence relations are distributive ([36], quoted as 2.3 and 2.5 below). Among such "congruence–distributive" equational classes are the class of all lattices and every equational class of lattices with finitely many additional operations. Jónsson found that membership in a congruence–distributive equational class entitles an algebra to privileges not usually enjoyed even by individual algebras with distributive congruence lattices. For example, if A is a finite algebra in a congruence–distributive equational class, then no subdirectly irreducible algebra of the same operational type, but with more elements than A, can have the same identities as A [23, 36]. Further relations between congruence–distributivity and identities were later developed, as explained in [4]. In this setting, McKenzie's theorem for lattices and lattices with added operations suggested the following theorem, which was announced in [3] and will be proved constructively in this paper.

1.3 THEOREM. *If a finite algebra A is a member of a congruence–distributive equational class of finite type, then A has a finite equational basis.*

(An equational class is said to be of finite type if it has only finitely many basic operations. In comparison, even a two-element lattice with infinitely many additional nullary operations certainly has no finite equational basis.)

This theorem contains McKenzie's theorem, and so applies to finite lattices and to such lattice-ordered structures as orthomodular lattices [11, 12, 33], Heyting algebras, [4, 56, 80], cylindric algebras of finite dimension [30, 31], and relation algebras [38, 76]. (See also [4, Theorem 2.15(d)].) The theorem also applies to such non-lattice-ordered structures as implication algebras [62], arithmetical algebras [70], and tournaments [24]. (Of course, the theorem does not apply to many of the other positive cases listed in 1.1.)

The proof offered in this paper is in essence the original one, based on a combinatorial calculation using Ramsey's theorem; efforts to find a shorter constructive proof have not yet borne fruit. The proof, although long, does provide some concomitant insight into the structural implications of congruence–distributivity.

Shorter, nonconstructive proofs of Theorem 1.3 do now exist: Herrmann has devised an ingenious nonconstructive shortcut in the original proof, for the case of finite lattices [32]. This method can be adapted to prove Theorem 1.3 (see Sections 9–11 below). Herrmann's method also yields finite-basis results for some other classes of lattices. Subsequently, Makkai [52] has shown how the theory developed in [4] can be used to give a short nonconstructive proof of the full version of Theorem 1.3. Makkai's method, while similar in spirit to Herrmann's, is different in detail and is even more efficient. The methods of Hermann and Makkai are analyzed and contrasted in Sections 9–11.

Interest in relations between identities and congruence relations has been growing rapidly. One attractive question is the possible generalization of Theorem 1.3 to congruence–modular equational classes, a feat that would at last unify the Oates–Powell theorem for groups, Kruse's theorem for rings, and the present congruence–distributive theory.

1.4. *Outline*

The main goal of the paper is a constructive proof of Theorem 1.3. The development is phrased, however, so that as much of the theory as possible is applicable to other problems. The paper concludes with a synthesis and generalization of the known nonconstructive proofs.

The sections are organized as follows. Section 2: Reduction of Theorem 1.3 to the consideration of classes defined by a cardinality restriction. Section 3: An exposition (with some proofs postponed) of "k-translations," the basic tool of the paper. Section 4: A finite-basis criterion, applicable to a wider class of problems. Sections 5, 6: Technical lemmas, and proofs previously deferred. Sections 7, 8: The application of the finite-basis criterion of Section 4 to the reduced problem of Section 2, via a combinatorial argument using Ramsey's theorem. Sections 9–11: An exposition and reformulation of the nonconstructive methods of Hermann and Makkai, in terms of finitely subdirect irreducibility.

This paper has been organized so that the basic framework of the reasoning may be ascertained from Sections 2, 3, 4, 7, and 9 alone. Technical details have been relegated to the remaining sections.

1.5. *Terminology and Notation*

E is invariably an equational class, **K** a subclass of **E**. The equational class generated by a class **K** is denoted by $\mathbf{K}^e$. The equational class generated by a single algebra A is denoted by A^e. An *equational basis for* **K** (or A) is a set of defining identities for $\mathbf{K}^e$ (or A^e). Thus an equational basis for **K** and an equational basis for $\mathbf{K}^e$ are the same thing. No notational distinction is made between an algebra and its underlying set. As mentioned above, an algebra A is said to be *of finite type* if A has only finitely many basic operations. A is said to be *subdirectly irreducible* (SI) if A has no nontrivial representation as a subdirect product, or equivalently, if there is a least nonzero congruence relation on A. A is said to be *finitely subdirectly irreducible* (FSI) if A has no nontrivial representation as a subdirect product of finitely many factors, or equivalently, if every two nonzero congruence relations on A have a nonzero intersection. (McKenzie [58], however, uses the abbreviation "FSI" to mean "finite and SI.") $A^{(2)}$ will denote the set of 2-element subsets of A (nontrivial unordered pairs formed by elements of A). The congruence relation in A generated by identifying a and b will be denoted by $\theta(a, b)$; such congruence relations are called "principal."

This paper is based in part on the ideas delineated in [2] and [4]. While these papers are on occasion quoted for motivational purposes, the present treatment is intended to be self-contained. Some additional papers with a bearing on congruence distributivity are listed in the references.

General references on lattice theory are [9, 16, 27, 75]; on universal algebra [15, 26, 37, 55, 68]; on ordered algebras [25]; on model theory [26, 73]; and on equational logic [59, 77].

2. Algebras of Bounded Cardinality

It is easy to describe a finite algebra A of finite type, up to isomorphism, by finitely many sentences. To pass from a description of A to the required equational description of A^e, we must produce, from such sentences describing A, some list of identities. Unfortunately, the describing sentences for A will contain existential quantification and negation and so lie, in structure, at some distance from identities.

It is easier to throw A into the pool of all algebras of the same or lesser cardinality, within an enveloping congruence–distributive equational class. Such a cardinality restriction has an easier syntactic form. The main theorem, then, will be derived in this section as a consequence of the following fact.

2.1 Theorem. *Let* $\mathbf{E}$ *be a finitely based congruence–distributive equational class of finite type. For any positive integer* m, *let* $\mathbf{E}_m$ *be the class of members of* $\mathbf{E}$ *of cardinality at most* m. *Then* ${\mathbf{E}_m}^e$ *is finitely based. More generally, if* $\mathbf{E}$ *is not assumed to be finitely based, then* ${\mathbf{E}_m}^e$ *is still finitely based "relative to* $\mathbf{E}$*," i.e., is definable by finitely many identities in addition to those defining* $\mathbf{E}$.

A constructive proof of this theorem will be the goal of the next several sections and will be completed only in Section 8. (Some additional discussion precedes 7.1.)

In order to derive Theorem 1.3 from Theorem 2.1, three facts are needed. The first is also used by McKenzie [57, Sect. 2].

2.2 THEOREM [7, Theorem 11, p. 442]. *Let A be a finite algebra of finite type. Then the set of m-variable identities of A has a finite basis, for any positive integer m.*

Birkhoff's proof is constructive; the identities are read off from the operation tables of A.

The second fact is a rephrasing of the key discovery of Jónsson [36, Theorem 3.3]. (The proof appears also in [26, p. 245].)

2.3 THEOREM [36]. *Let $\mathbf{K}$ be a class of algebras closed under formation of ultraproducts, subalgebras, and homomorphic images. If $\mathbf{K}^e$ is congruence-distributive, then every subdirectly irreducible* (SI) *algebra in $\mathbf{K}^e$ already lies in $\mathbf{K}$. More generally, the same is true of finitely subdirectly irreducible* (FSI) *algebras.*

The third fact:

2.4 PROPOSITION. *Any congruence-distributive equational class $\mathbf{E}$ is contained in a finitely based congruence-distributive equational class $\mathbf{E}'$ of the same type.*

This proposition is an immediate consequence of the following theorem of Jónsson, which characterizes congruence-distributivity by the existence of finitely many polynomials subject to finitely many identities. To prove Proposition 2.4, then, we merely choose such polynomials for $\mathbf{E}$ and use their identities to define $\mathbf{E}'$.

2.5 THEOREM [35, Theorem 2.1]. *An equational class $\mathbf{E}$ is congruence-distributive if and only if there exists an integer $n \geqslant 2$ and ternary polynomial expressions $t_0, \ldots, t_n$ such that the following identities hold for all algebras in $\mathbf{E}$.*

$$t_j(x, y, x) = x \qquad (j = 0, 1, \ldots, n); \tag{2.5a}$$

$$\begin{aligned} t_j(x, x, z) &= t_{j+1}(x, x, z) \qquad \text{if } j \text{ is even}; \\ t_j(x, z, z) &= t_{j+1}(x, z, z) \qquad \text{if } j \text{ is odd}; \end{aligned} \tag{2.5b}$$

$$t_0(x, y, z) = x, \qquad t_n(x, y, z) = z. \tag{2.5c}$$

(Observe that t_0 and t_n are included merely as a notational convenience.)

2.6. *Proof of Theorem* 1.1 *with Theorem* 2.1 *Assumed*

For $A \in \mathbf{E}$ as given, let m be the cardinality of A and let $\mathbf{E}_m$ be as in Theorem 2.1. We proceed by showing that (a) without loss of generality, $\mathbf{E}$ is determined by finitely many identities, (b) $\mathbf{E}_m{}^e$ is then determined by adding finitely many more identities and (c) A^e is determined by again adding finitely many more. Here (a) follows from Proposition 2.4, since $\mathbf{E}$ can be replaced by $\mathbf{E}'$ if necessary, and (b) is the assertion of Theorem 2.1. For (c), observe first that $\mathbf{E}_m$ satisfies the requirements for $\mathbf{K}$ in Theorem 2.3. Because any equational class is generated by its SI members, it follows that each equational subclass of $\mathbf{E}_m{}^e$ is generated by members lying in $\mathbf{E}_m$. Therefore A^e is determined, relative to $\mathbf{E}_m{}^e$, by any collection of identities sufficient to decide membership in A^e for members of $\mathbf{E}_m$ alone. The set of all m-variable identities of A is certainly one such collection; the finite basis of m-variable identities provided by Birkhoff (Theorem 2.2 above) is then another such collection.

Observe that this proof is constructive, if the Jónsson polynomials $t_0, \ldots, t_n$ are known—a condition invariably fulfilled in practice. In fact, it will always be tacitly assumed that each congruence–distributive equational class $\mathbf{E}$ discussed in this paper comes with a designated list of polynomial expressions $t_0, \ldots, t_n$.

3. k-Translations

Both the original Theorem 1.3 and its reduction (Theorem 2.1) to classes of bounded cardinality involve problems of this form: For a given class $\mathbf{K}$ of algebras, find a finite equational basis for $\mathbf{K}^e$.

For many such classes $\mathbf{K}$, including the case of the class $\mathbf{E}_m$ of Section 2, $\mathbf{K}^e$ was described in [4], first by a condition involving congruence relations and thence by an infinite set of identities. Even if a finite set of identities does exist for a particular $\mathbf{K}$, the method of that paper, in effect, does not know when to stop producing identities. The approach of the present paper is to locate the stopping point by tying congruence relations to something that can be counted. This counting mechanism is provided by an examination of certain translations in algebras.

The word "translation" for algebraic systems does not have a meaning agreed upon by all authors [9, p. 137; 28; 53] but all relevant usages coincide in one respect: A translation is some kind of unary algebraic function, i.e., function on an algebra A to itself, obtained by freezing all entries except one in a polynomial function of A [26, pp. 45, 54]. If φ is any such function, then $a \equiv b \pmod{\theta}$ implies $\varphi(a) \equiv \varphi(b) \pmod{\theta}$, for any congruence relation θ of A. Thus such functions can be used to ferry congruence information around in A. We shall define, for $k = 0, 1, 2, \ldots$, the concept of a "k-translation"—a map that carries congruence information at most a "distance" k in A.

This section will be devoted to a definition and discussion of the basic properties of k-translations. Their application to a description of classes $\mathbf{K}^e$ will be left to the next section.

3.1 Definition. Let A be an algebra in a congruence-distributive equational class $\mathbf{E}$ with chosen Jónsson polynomials $t_0, \ldots, t_n$. By a *basic translation* on A let us means a unary algebraic function obtained by freezing all entries, except one, in any of the basic operations of A [53, Sect. 1]. By a *Jónsson-polynomial translation* on A let us mean a unary algebraic function obtained by freezing two entries of one of the ternary polynomials t_i. Now for $k = 0, 1, \ldots$ let us say that a function φ on $A \to A$ is a *k-translation* if φ can be expressed as a composition of at most k functions, each of which is a basic translation or a Jónsson-polynomial translation. (The identity function is to be considered a 0-translation.)

For example, if L is a lattice and $a \in L$, then $m_a : L \to L$ given by $m_a(x) = a \wedge x$ is a basic translation, and $\varphi: L \to L$ given by $\varphi(x) = ((x \vee a) \wedge b) \vee c$, for fixed a, b, c, is a 3-translation.

The inclusion of Jónsson-polynomial translations in Definition 3.1 is a technical convenience that greatly simplifies statements of some theorems to come. For the case of lattices, the use of Jónsson-polynomial translations can be avoided without difficulty, as in [2].

The k-translations (for varying k) form a Mal'cev family in the sense of [4, Remark 3.10] (see 5.1).

It will also be convenient to have a simple notation for the movement in an algebra A produced by applying a k-translation:

Any map on $A \to A$ induces a map on various other sets to themselves (the set of ordered pairs from A, the set of unordered pairs from A, the set of all subsets of A, the set of all sequences from A, etc.). If ξ, η are elements of A or of any one of these sets, let us write $\xi \to_k \eta$ if there is a k-translation that takes ξ to η. Let us write $\xi \to \eta$ if $\xi \to_k \eta$ for some k.

For example, if a/b is a formal quotient in a lattice L and c is any element, then $a/b \to_1 (a \vee c)/(b \vee c)$. More generally, in a lattice, if c/d is weakly projective into a/b then $a/b \to c/d$. (Cf. [2, Lemma 3.2]. The converse statement would hold were it not for the presence of Jónsson-polynomial translations in the picture.)

As another example, the remark that k-translations preserve congruence in an algebra A can be rephrased: If $\{a, b\} \twoheadrightarrow \{c, d\}$, then $\theta(a, b) \supseteq \theta(c, d)$. An application: In [4] equations such as $\theta(a_1, b_1) \cup \cdots \cup \theta(a_N, b_N) = 0$ were fundamental. A sufficient condition that this equation *fail*, then, is that there exist $c \neq d$ such that $\{a_i, b_i\} \twoheadrightarrow \{c, d\}$ for all i. In other words, if we restrict our attention to the set $A^{(2)}$ of nontrivial (two-element) unordered pairs from A and regard $\twoheadrightarrow$ as a quasi-order on $A^{(2)}$, then the necessary condition for failure is that the N pairs $\{a_i, b_i\}$ in $A^{(2)}$ have a common upper bound in $A^{(2)}$ (cf. [2, Lemma 3.2]).

An interesting fact is that this necessary condition is also sufficient. Because this fact is crucial to the present paper, it is appropriate to give it a name, the "arrow lemma":

3.2 LEMMA (Arrow Lemma). *Let* $\mathbf{E}$ *be a congruence–distributive equational class and let* $A \in \mathbf{E}$. *Then for* $a_1, b_1, \ldots, a_N, b_N \in A$ *with* $a_i \neq b_i$ *the following are equivalent.*

(1) $\bigcap_i \theta(a_i, b_i) > 0$,

(2) *the pairs* $\{a_i, b_i\}$ *have a common upper bound in* $A^{(2)}$, *under* $\twoheadrightarrow$.

The proof, which is somewhat technical, will be given in Proof 5.2. The proof does depend on the incorporation of the polynomials t_i in the definition of a k-translation.

3.3 COROLLARY. *An algebra A in a congruence–distributive equational class* $\mathbf{E}$ *is finitely subdirectly irreducible* (FSI) *if and only if the quasi-ordered set* $(A^{(2)}, \twoheadrightarrow)$ *is directed.*

Proof. A is FSI if and only if the intersection of any two nonzero congruence relations is nonzero. Since every congruence relation is a join of principal ones, this condition can be checked by testing intersections of principal congruence relations alone. The case $N = 2$ of Lemma 3.2 then applies.

While Corollary 3.3 applies in particular to SI algebras, there does not seem to be a simple characterization of SI algebras in terms of the quasi-ordered set $(A^{(2)}, \twoheadrightarrow)$.

3.4 *Terminology for Future Reference.* Suppose $\{a_1, b_1\}, \ldots, \{a_N, b_N\}$

have a common bound $\{c, d\}$ in $A^{(2)}$, reachable via k-translations for some particular integer k. In other words, $\{a_i, b_i\} \to_k \{c, d\}$ for each i. Then let us say simply, "$\{a_1, b_1\},..., \{a_N, b_N\}$ are k-bounded" in $A^{(2)}$, with "k-bound" $\{c, d\}$.

4. A Finite-Basis Criterion

For any positive integer m, the property of having cardinality at most m can be described by the sentence μ_m given by $\mu_m = (\forall x_1,..., x_{m+1})\mathbf{OR}_{i<j} x_i = x_j$, where **OR** denotes disjunction. This sentence is an example of a "universal disjunction of equations" (UDE) [2, 4], i.e., a sentence δ of the form

$$\delta = (\forall x_1,..., x_e) f_1(\mathbf{x}) = g_1(\mathbf{x}) \mathbf{OR} \cdots \mathbf{OR} f_N(\mathbf{x}) = g_N(\mathbf{x}), \tag{4.0}$$

where the f_i, g_i are polynomial expressions. (We shall always assume such expressions are in the language of **E**.)

Thus the assertion of Theorem 2.1, that $\mathbf{E}_m{}^e$ is finitely based, is an answer, in one instance, to a larger question: Let **E** be a finitely based congruence–distributive equational class, let δ be a UDE formed from polynomial expressions of **E**, and let **K** be the class of members of **E** that satisfy δ. When is $\mathbf{K}^e$ finitely based?

In this section, Criterion 4.5 will answer this question. In Section 6, the criterion will be applied to yield a constructive proof of Theorem 2.1, and thereby of the main Theorem 1.3. The criterion will also yield a short nonconstructive proof of Theorem 2.1 in Sections 9–11, in the form of a simultaneous generalization (Theorem 9.1) of the theories of Herrmann and of Makkai [32, 52].

4.1. *Descriptions of* $\mathbf{K}^e$

An obvious starting point in developing such a criterion is simply to describe $\mathbf{K}^e$. This was the purpose of [4]. One description had in essence been given in [36]: $\mathbf{K}^e$ consists of those members of **E** that satisfy δ residually, i.e., are subdirect products of algebras satisfying δ in the ordinary sense (models of δ). In [4], two more descriptions were given. Of these, the second was simply a set of identities, $\mathbf{I}[\delta]$. The first, on which the second depended, can be summarized (in a terminology helpful for the present paper) as follows.

(a) If δ fails in A, then there exist $c_1, \ldots, c_l \in A$ such that $f_i(\mathbf{c}) \neq g_i(\mathbf{c})$ for all i. Let us concentrate not on the elements c_j but rather on the pairs $\{f_i(\mathbf{c}), g_i(\mathbf{c})\}$, which we may call a set of *failure pairs* for δ. (b) For pairs of elements $\{a_1, b_1\}, \ldots, \{a_N, b_N\}$ in an algebra A, let us call the congruence relation $\bigcap_i \theta(a_i, b_i)$ their *congruence intersection*. (c) A satisfies δ *primitively* [4, 32] if, in our present terminology, the congruence intersection of any set of failure pairs for δ in A is 0. Then $\mathbf{K}^e$ consists of all algebras in $\mathbf{E}$ that satisfy δ primitively.

For example, if $\mathbf{2} = \{0, 1\}$ is the two-element chain, then $\mathbf{2} \times \mathbf{2}$ fails to satisfy the cardinality sentence μ_2, but does satisfy μ_2 primitively.

A glance at Lemma 3.2 shows that primitive satisfaction of δ can be reexpressed in terms of $\twoheadrightarrow$-boundedness to yield the following equivalences.

4.2 THEOREM. *Let* $\mathbf{E}$ *be a congruence–distributive equational class, let* δ *be a* UDE, *and let* $\mathbf{K}$ *be the class of models of* δ *in* $\mathbf{E}$. *Then the following conditions are equivalent for* $A \in \mathbf{E}$.

(1) $A \in \mathbf{K}^e$,

(2) *A satisfies* δ *residually,*

(3) *A satisfies* δ *primitively, i.e., the congruence intersection of any set of failure pairs for* δ *in A is* 0,

(4) *A satisfies the set of identities* $\mathbf{I}[\delta]$,

(5) *no set of failure pairs for* δ *in A is bounded in* $A^{(2)}$, *under* $\twoheadrightarrow$.

4.3. *The Question of Finite Axiomatizability*

How can a criterion for the finite axiomatizability of $\mathbf{K}^e$ be drawn from Theorem 4.2? The essential observations needed are the following. ($\mathbf{E}$ must be of finite type; for simplicity, let us suppose also that $\mathbf{E}$ is finitely based.) (a) The equivalence of (1) and (5) can be reexpressed by saying that $\mathbf{K}^e$ is determined (relative to $\mathbf{E}$) by this infinite list of assertions, where $k = 0, 1, 2, \ldots$: "No set of failure pairs for δ is k-bounded." (b) Each such assertion is expressible by a first-order sentence. (c) Therefore, according to Gödel's compactness theorem, $\mathbf{K}^e$ is already determined by finitely many of these assertions if and only if $\mathbf{K}^e$ is already determined by finitely many of its defining identities. (d) Since the assertions become stronger with increasing k, any finite set of them is equivalent to a single one of them. (e) Thus $\mathbf{K}^e$ is finitely based if and only if there exists k_0 such that the k_0th assertion determines $\mathbf{K}^e$. (f) Since, as noted, $\mathbf{K}^e$ is determined by *all* the assertions, $\mathbf{K}^e$ is finitely based if and only if it meets this condition (phrased in the contrapositive): (*) There exists k_0 such that if an algebra $A \in \mathbf{E}$ has any bounded set of failure pairs for δ, then A already has a k_0-bounded set of failure pairs, i.e., a set of failure pairs bounded in $A^{(2)}$ via k_0-translations (see (3.4)).

The condition (*) represents a criterion of sorts. It could stand improvement in two respects: (i) It has a conditional clause (the existence of some bounded set of failure pairs) that may be difficult to verify, and (ii) even when the condition (*) is met, the reasoning given does not produce an explicit list of defining identities for $\mathbf{K}^e$.

Objection (i) will be met by showing that attention can be restricted to SI algebras, where, according to Corollary 3.3, *any* set of pairs in $A^{(2)}$ is bounded. Objection (ii) will be met by proving a relation between the k_0 of (*) and an explicit set of defining identities for $\mathbf{K}^e$, similar to the identities constructed in [4, Sect. 4].

Because the construction and validity of these identities depend on the technical lemmas relegated to Section 5, it will be clearest now merely to codify their existence and properties, for reference:

4.4 THEOREM. *Let* $\mathbf{E}$ *be a congruence-distributive equational class of finite type, and let* δ *be a* UDE (*in the language of* $\mathbf{E}$). *Then there exists for each* k, $k = 0, 1, \ldots$, *a set* $\mathbf{I}_k[\delta]$ *of identities, such that*

(a) *the union of the sets* $\mathbf{I}_k[\delta]$ *determines* (*within* $\mathbf{E}$) *the equational class generated by those members of* $\mathbf{E}$ *that satisfy* δ;

(b) $\mathbf{I}_k[\delta]$ *is finite for each* k;

(c) $\mathbf{I}_k[\delta] \subseteq \mathbf{I}_{k+1}[\delta]$ *for each* k;

(d) *if* $A \in \mathbf{E}$ *contains a k-bounded set of failure pairs for* δ, *then some member of* $\mathbf{I}_k[\delta]$ *fails in* A; *and*

(e) *if some member of* $\mathbf{I}_k[\delta]$ *fails in* $A \in \mathbf{E}$, *then* A *contains a* $(k + 2N)$-*bounded set of failure pairs for* δ, *where* N *is the number of disjugands in* δ.

The construction and proof will be given in Proof 6.3, after the machinery of Section 5 is available. The property most relevant for our present purposes is (d).

Here, then, is the criterion.

4.5 CRITERION. *Let* **E** *be a congruence–distributive equational class of finite type, let* δ *be a* UDE *formed from* **E**-*polynomial expressions, and let* **K** *be the class of models of* δ *in* **E**. *Then the following statements are equivalent.*

(1) $\mathbf{K}^e$ *is finitely based (relative to* **E**).

(2) $\mathbf{K}^e$ *is determined, relative to* **E**, *by the finitely many identities in* $\mathbf{I}_k[\delta]$, *for some* k.

(3) *There is an integer* k' *such that whenever an algebra* $A \in \mathbf{E}$ *has a set of failure pairs for* δ *that is bounded in* $A^{(2)}$, *then* A *already has a* k'-*bounded set of failure pairs for* δ;

(3_{SI}) *There is an integer* k'' *such that when* δ *fails in an* SI *algebra* $A \in \mathbf{E}$, *then* A *has a* k''-*bounded set of failure pairs for* δ.

Moreover, if a suitable value of one of the parameters k, k', k'' *is known, then suitable of the other two parameters can be found in terms of the number* N *of disjugands of* δ, *as follows. Given* k, *choose* $k' = k'' = k + 2N$, *given* k', *choose* $k = k'' = k'$; *given* k'', *choose* $k = k''$, $k' = k'' + 2N$.

The proof appears as 4.7 below.

4.6 *Remarks.* (a) For the definition of "k-bounded," see 3.4. (b) Of course, if **E** itself is finitely based, the qualification "relative to **E**" in condition (1) can be dropped. (c) The criterion generalizes to the case where **K** is defined, relative to **E**, by finitely many UDE's; $\mathbf{K}^e$ will be finitely based (relative to **E**) if the condition (3) or (3_{SI}) is met for each UDE individually. (d) The condition (3_{SI}) could be replaced by an equivalent condition (3_{FSI}). (e) In conditions (3) and (3_{SI}), it is *not* asserted that *any* given set of failure pairs is k'-bounded or k''-bounded. (f) The criterion will usually be applied in the form (3_{SI}) $\Rightarrow$ (2). (g) The knowledge that k can be chosen to be k'' makes the criterion constructive, if k'' is known. (h) For equational classes of lattices, the criterion holds even if Jónsson-polynomial translations are omitted in the definition of $\rightarrow_k$ and k-boundedness.

4.7 *Proof of the Criterion, in the Order* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (S_{SI}) $\Rightarrow$ (1). (1) $\Rightarrow$ (2): Use Gödel's compactness theorem, relativized to **E**. (2) $\Rightarrow$ (3): If k is known and A has a bounded set of failure pairs for δ, then by Theorem 4.4(a), (d), $A \notin \mathbf{K}^e$. Then by (2), some identity in $\mathbf{I}_k[\delta]$ fails in A. By (e) of Theorem 4.4, A has a $(k + 2N)$-bounded set of failure pairs for δ, so (3) holds with $k' = k + 2N$. (3) $\Rightarrow$ (3_{SI}): By Corollary 3.3, any set of failure pairs in an SI algebra is bounded, so the assertion of (3) reduces to the assertion of (3_{SI}), with $k'' = k'$. $(3_{SI}) \Rightarrow (1)$: Because $\mathbf{I}_{k''}[\delta]$ is finite and already forms *part* of a defining set of identities for $\mathbf{K}^e$, it is enough to show that the equational class determined by $\mathbf{I}_{k''}[\delta]$ is no larger than $\mathbf{K}^e$. In fact, because any equational class is generated by its SI members, it is enough to show that any SI model of $I_{k''}[\delta]$ is in $\mathbf{K}^e$. But (3_{SI}) and Theorem 4.4(d) combine to say: If δ fails in an SI algebra, then so does $\mathbf{I}_{k''}[\delta]$. In other words, every SI model of $I_{k''}[\delta]$ is even in $\mathbf{K}$. This completes the proof of the equivalence of (1)–(3_{SI}). With regard to the relations between choices of k, k', k'', the only nontrivial relations are implicit in the proof of equivalence just concluded.

(Note: The summary (Theorem 4.2) of facts from [4] was in the end supplanted in the proof by a use of Theorem 4.4, whose proof in Proof 6.3 will likewise not depend on Theorem 4.2. Thus the reasoning of this paper is actually independent of [4].)

5. The Sequence Lemmas

This section provides the calculations with Jónsson's polynomials t_i that are necessary to supply the proof of Lemma 3.2, the "arrow lemma." The resulting facts will be used again in Section 6, in conjunction with a study of the sets $\mathbf{I}_k[\delta]$ of identities, and in Sections 7–8, where they are at the core of the main computation of this paper.

As usual, let us work inside a congruence–distributive equational class $\mathbf{E}$ with specified polynomial expressions t_j.

The following terminology will be helpful. For elements c, d of an algebra $A \in \mathbf{E}$, a finite sequence $S = \langle c_0, c_1, ..., c_r \rangle$, where $c_0 = c$ and $c_r = d$, will be called simply a "sequence from c to d"; the unordered pairs $\{c_{i-1}, c_i\}$ $(1 \leqslant i \leqslant r)$ will be called the "links" of S. The possibility $c_{i-1} = c_i$ is admitted (a "trivial link").

In this terminology, for example, Mal'cev's construction of principal congruence relations, adapted to translations, can be expressed as follows.

5.1 Proposition (Mal'cev [53; 26, p. 54]). *For elements a, b, c, d of an algebra A in the congruence–distributive equational class $\mathbf{E}$, $c \equiv d \bmod \theta(a, b)$ if and only if there is a sequence from c to d whose links are k-translates of $\{a, b\}$ for some k.*

(The proof is a straightforward verification that the given requirement on pairs $\langle c, d\rangle$ determines a congruence relation with the properties of $\theta(a, b)$. The inclusion of Jónsson-polynomial translations in the definition of k-translation is not needed here.)

Proposition 5.1 suggests the following reasoning in support of Lemma 3.2, the "arrow lemma," whose proof has remained on the agenda.

5.2 *Proof of Lemma* 3.2. (2) $\Rightarrow$ (1): If $\{c, d\}$ is a common upper bound of $\{a_1, b_1\},..., \{a_N, b_N\}$ in $A^{(2)}$, then $c \neq d$ by the definition of $A^{(2)}$, and $c \equiv d \bmod \theta(a_i, b_i)$ for each i by the fact that $\rightarrow$ preserves congruence. Thus (1) holds. (1) $\Rightarrow$ (2): If $\{a_1, b_1\},..., \{a_N, b_N\}$ have a nonzero congruence intersection, then choose $c \neq d$ such that $c \equiv d \bmod \theta(a_i, b_i)$ for each i. Thus $\{c, d\} \in A^{(2)}$, but $\{c, d\}$ may not be a common upper bound of the $\{a_i, b_i\}$ in $A^{(2)}$. Nevertheless, by Proposition 5.1 there are N sequences $S_1,..., S_N$ from c to d such that each link of S_i is a k-translate of $\{a_i, b_i\}$, for some k. The proof will be complete if this last sentence can be supplied: It is possible to choose one link from each S_i so that the N chosen links are nontrivial (are in $A^{(2)}$) and have a common upper bound in $A^{(2)}$ via $\rightarrow$.

This last sentence is justified by the following lemma, which gives even more precise information. (Again, a name is provided for future convenience. The proof is given in 5.6.)

5.3 Lemma (Multisequence lemma). *Let $S_1,..., S_N$ be sequences all joining the same two distinct elements c, d in an algebra A of the congruence-distributive equational class* **E**. *Then it is possible to choose, for each i, one link of S_i, to be called its "key link," so that the N key links are nontrivial and are N-bounded in $A^{(2)}$.*

The knowledge that the bound can be reached from each key link in at most N steps will be crucial in future applications of the lemma.

5.4 *Discussion.* For the special case where **E** is an equational class of lattices, one can usually restrict attention to the case where $c < d$ and the sequences $S_1,..., S_N$ are ascending chains. A direct proof of the Multisequence lemma is then easy: Write $S_j = \langle c_0^{(j)},..., c_{r(j)}^{(j)}\rangle$. Among N-tuples $\mathbf{i}$ of indices with the property $c_{i(1)}^{(1)} \wedge \cdots \wedge c_{i(N)}^{(N)} > c$, choose one (again called $\mathbf{i}$) that is minimal with respect to this property under the coordinatewise ordering of N-tuples. If $e = c_{i(1)}^{(1)} \wedge \cdots \wedge c_{i(N)}^{(N)}$, then the 1-translation $\varphi(x) = x \wedge e$ carries the $i(j)$th link of S_j to $\{c, e\}$ for each j. Thus the N key links are not merely N-bounded, but 1-bounded.

For the proof of the Multisequence lemma in full generality, though, one must use an induction requiring some preparation (Lemma 5.5). Jónsson's polynomials $t_0, \ldots, t_n$, not needed for lattices, now enter through the following construction (cf. [4, proof of Lemma 3.5]).

Let $S = \langle c_0, c_1, \ldots, c_r \rangle$ ($c_0 = c$, $c_r = d$) be a sequence from c to d in $A \in \mathbf{E}$. The *derived* sequence of S, denoted S^t, is a certain new sequence from c to d: S^t is obtained by stringing together $n - 1$ segments. The first segment runs from $c = t_1(c, c, d)$ to $t_1(c, d, d) = t_2(c, d, d)$; specifically, its elements are $t_1(c, c_i, d)$ for $i = 0, 1, \ldots, r$. The second segment runs from $t_2(c, d, d)$ to $t_2(c, c, d) = t_3(c, c, d)$; its elements are $t_2(c, c_i, d)$ for $i = r, r - 1, \ldots, 0$. The sequence S^t continues with elements $t_j(c, c_i, d)$, for i alternately increasing and decreasing between successive changes in j. The last segment ends at d, in the guise of either $t_{n-1}(c, c, d)$ or $t_{n-1}(c, d, d)$, depending on the parity of n. (S^t is related to [36, Sect. 2, inclusion (4)].)

This construction is used in proving the following lemma, which isolates the inductive step of the Multisequence lemma.

5.5 Lemma (Single-sequence Lemma). *Let S be a sequence between distinct elements c, d in an algebra A of the congruence-distributive equational class* $\mathbf{E}$. *Then there is a nontrivial link of S such that this link and $\{c, d\}$ are 1-bounded in $A^{(2)}$. A bound is in fact the first nontrivial link of the derived sequence S^t.*

Proof. Let $e = t_j(c, c_i, d)$ be the first term of S^t that is different from c. Suppose i was increasing during the jth segment of S^t; the case of decreasing i is similar. Then the preceding term is $t_j(c, c_{i-1}, d) = c$. Thus $\{c, e\} = \{t_j(c, c_{i-1}, d), t_j(c, c_i, d)\}$, so that $\{c_{i-1}, c_i\} \rightarrow_1 \{c, e\}$ via a middle-entry Jónsson-polynomial translation; further, $\{c, e\} = \{t_j(c, c_i, c), t_j(c, c_i, d)\}$, so that $\{c, d\} \rightarrow_1 \{c, e\}$ via a third-entry Jónsson-polynomial translation.

5.6 *Proof of Lemma* 5.3, *the Multisequence Lemma.* Rather than use the statement of the lemma itself as an inductive hypothesis, it is convenient to use a stronger hypothesis: (*) The N key links can be chosen so that $\{c, d\}$ and the N key links are together N-bounded. For (*), the case $N = 0$ makes sense and is trivially true. Next, assume (*) for $N = M$ and consider the case of $N = M + 1$ sequences. By (*), then, $\{c, d\}$ and M links from $S_1, \ldots, S_M$ have a common M-bound in $A^{(2)}$, say $\{c', d'\}$. The M-translation φ taking $\{c, d\}$ to $\{c', d'\}$ carries S_{M+1} to a sequence S'_{M+1} from c' to d'. According to the "single-sequence lemma" Lemma 5.5, there is a common 1-bound $\{c'', d''\}$ for $\{c', d'\}$ and some link of S'_{M+1} representable as φ applied to a link of S_M. $\{c'', d''\}$ is thus an $(M + 1)$-translate of that link of S_M and is also an $(M + 1)$-translate of $\{c, d\}$ and of the chosen links of $S_1, \ldots, S_M$, by an arrow $\rightarrow_{M+1}$ routed through $\{c', d'\}$. The case $N = M + 1$ is therefore verified, and the induction is complete.

6. Construction of Identities

For $\mathbf{E}$, δ, and $\mathbf{K}$ as usual, Theorem 4.4 asserted the existence of a defining set of identities for $\mathbf{K}^e$ in which the identities are closely tied to k-translations for various k. This section is devoted to the requisite construction of identities and consequent proof of Theorem 4.4.

In [4, Sect. 4], identities determining $\mathbf{K}^e$ were likewise given. The identities given here will differ mainly in one small but essential respect: The "composition of pairs" by association from the right instead of association from the left. The resulting new identities are more easily studied using k-translations, whereas the former identities were better suited to study by intersection of principal congruence relations. The lattice identities constructed in [2, Sect. 3] are more nearly of the present form. In fact, the reasoning of this section is a generalization of the reasoning of [2, Proof of Theorem 3.1]. [In that proof, note one misprint: In the next-to-last sentence, read $\notin$ for $\in$.]

A fixed congruence-distributive equational class $\mathbf{E}$ of finite type with known Jónsson-polynomial expressions $t_1, \ldots, t_{n-1}$ will be assumed. Let δ be a fixed UDE in the usual form (4.0), and let $\mathbf{K}$ be the subclass of $\mathbf{E}$ determined by δ.

Let us borrow the notational conveniences of [4, Sections 3, 4]: (1) For $A \in \mathbf{E}$ and $\langle a, b\rangle, \langle c, d\rangle \in A \times A$ and $j = 1, \ldots, n-1$, let $\langle a, b\rangle *_j \langle c, d\rangle$ denote the pair $\langle t_j(a, c, b), t_j(a, d, b)\rangle$ obtained by applying the 1-translation $\varphi(x) = t_j(a, x, b)$ to c and d. Similarly, if f_1, g_1, f_2, g_2 are polynomial expressions, let $\langle f_1, g_1\rangle *_j \langle f_2, g_2\rangle$ denote the pair of composite polynomial expressions $\langle t_j(f_1, f_2, g_1), t_j(f_1, g_2, g_1)\rangle$. (2) If f, g are polynomial expressions, let $\mathrm{equ}(f, g)$ denote the equation $f = g$.

Next, choose for each k a family $\mathscr{P}^{(k)}$ of polynomial expressions inducing all k-translations. This can be done economically and canonically, as follows. Each polynomial expression will involve variables $x_0, x_1, \ldots$ as needed. Let $\mathscr{P}^{(0)}$ consist only of x_0. Let each member of $\mathscr{P}^{(k+1)}$ be either (i) a member of $\mathscr{P}^{(k)}$; (ii) a composition $t_j(p, x_{r+1}, x_{r+2})$

or $t_j(x_{r+1}, p, x_{r+2})$ or $t_j(x_{r+1}, x_{r+2}, p)$, where $p \in \mathscr{P}^{(k)}$ and variables $x_0, x_1, \ldots, x_r$ appear in p; or (iii) a similar composition using a basic-operation expression of any "arity" > 0 in place of t_j. Thus x_0 is nested innermost.

Finally, for each k let $\mathbf{I}_k[\delta]$ be the set of all identities of the form

$$(\forall \mathbf{x})(\forall \mathbf{W}) \operatorname{equ}(\langle F_1, G_1\rangle *_{j(2)} \cdots *_{j(N)} \langle F_N, G_N\rangle), \tag{6.1}$$

where (a) for each i, a polynomial expression $p_i \in \mathscr{P}^{(k)}$ of any arity $e(i)$ is chosen and $F_i = p_i(f_i(\mathbf{x}), w_1{}^i, \ldots, w^i_{e(i)-1})$, $G_i = p_i(g_i(\mathbf{x}), w_1{}^i, \ldots, w^i_{e(i)-1})$; (b) $j(2), \ldots, j(N) \in \{1, \ldots, n-1\}$; (c) $\mathbf{W}$ denotes the list of all variables $w_j{}^i$ used; and (d) the operations $*_{j(i)}$ are understood to be associated from the *right*.

In preparation for the proof that these identities do fulfill the requirements of Theorem 4.4, the following lemma contains all the necessary facts about iterated $*_j$ operations on pairs.

6.2 LEMMA. *For* $\mathbf{E}$ *as above,* $A \in \mathbf{E}$, *and* $\langle a_1, b_1\rangle, \ldots, \langle a_N, b_N\rangle \in A \times A$, *the set* H *of pairs of the form* $\langle a_1, b_1\rangle *_{j(2)}(\langle a_2, b_2\rangle *_{j(3)}(\cdots *_{j(N)}\langle a_N, b_N\rangle)\cdots))$, *where* $j(2), \ldots, j(N) \in \{1, \ldots, n-1\}$ *has these properties*:

(i) *If* $a_i = b_i$ *for some* i, *then* $r = s$ *for all* $\langle r, s\rangle \in H$.

(ii) *If the unordered pairs* $\{a_i, b_i\}$ *are all equal to the same pair* $\{c, d\}$ *with* $c \neq d$, *then there is at least one pair* $\langle r, s\rangle \in H$ *with* $r \neq s$.

(iii) *If there is at least one pair* $\langle r, s\rangle \in H$ *with* $r \neq s$, *then the unordered pairs* $\{a_1, b_1\}, \ldots, \{a_N, b_N\}$ *are* $2N$-*bounded in* $A^{(2)}$.

In (iii), observe that $\{a_i, b_i\} \in A^{(2)}$ for each i, by (i). This lemma will be proved as 6.4 below.

6.3 *Proof of Theorem* 4.4 *using Lemma* 6.2. Observe that (b) and (c) hold by construction. (a) Suppose that A satisfies δ, i.e., $A \in \mathbf{K}$. Then for any identity σ in $\mathbf{I}_k[\delta]$ and any evaluation of $\mathbf{x}$, $\mathbf{W}$, the values of f_i, g_i coincide for some i, and so the values of F_i, G_i coincide as well. Then Lemma 6.2 (i) applies to show that σ holds. Thus the equational subclass $\mathbf{E}_0$ of $\mathbf{E}$ determined by $\bigcup_k \mathbf{I}_k[\delta]$ contains $\mathbf{K}$ and hence $\mathbf{K}^e$. To complete the proof, we must check that $\mathbf{E}_0$ is no larger than $\mathbf{K}^e$. Because each equational class is determined by its SI members, it is enough to check that an SI algebra A *not* in $\mathbf{K}^e$ fails to satisfy $\mathbf{I}_k[\delta]$ for some k. But $A \notin \mathbf{K}^e$ implies $A \notin \mathbf{K}$, so that A contains some set of failure pairs for δ. Since A is SI, these failure pairs are k-bounded for some k, by Corollary 3.3. That $\mathbf{I}_k[\delta]$ then must fail is precisely the assertion of (d), which we now prove: (d) Let $\{c_1, d_1\}, \ldots, \{c_N, d_N\}$ be the failure pairs and let $\{c, d\}$ be their k-bound. Then for a suitable evaluation of $\mathbf{x}$ in A, $f_i(\mathbf{x})$ and $g_i(\mathbf{x})$ have values c_i, d_i for each i and for a suitable choice of the k-translation-inducing polynomial expressions $p_1, \ldots, p_N \in \mathscr{P}^{(k)}$ and suitable evaluation of $\mathbf{W}$ in A, F_i and G_i have values c, d (in either order) for each i. Then by Lemma 6.2 (ii), the $j(2), \ldots, j(N)$ can be chosen so that the corresponding identity in $\mathbf{I}_k[\delta]$ fails. (e): If some member of $\mathbf{I}_k[\delta]$ fails in A, i.e., for some particular evaluation of $\mathbf{x}$, $\mathbf{W}$ in A, let c_i, d_i be the corresponding values of f_i, g_i, and let a_i, b_i be the corresponding values of F_i, G_i, for each i. Then by the construction of $\mathscr{P}^{(k)}$, $\{c_i, d_i\} \to_k \{a_i, b_i\}$ for each i, and by Lemma 6.2 (iii), the $\{a_i, b_i\}$ are $2N$-bounded. Hence $\{c_1, d_1\}, \ldots, \{c_N, d_N\}$ are a $(k + 2N)$-bounded set of failure pairs for δ in A.

6.4 *Proof of Lemma* 6.2. (i) If $a_i = b_i$ for some i, then by property (2.5a) of the t_j, $\langle a_i, b_i\rangle *_j \langle c, d\rangle$ is a trivial (i.e., a diagonal) pair for any $\langle c, d\rangle$ and any j. Further leftward applications of $*$-operations, ending with $\langle a_1, b_1\rangle$, leave a trivial pair at each stage. (ii) Regard $\{a_N, b_N\}$ as a sequence from c to d (or d to c). Since $\{a_{N-1}, b_{N-1}\}$ also equals $\{c, d\}$, the elements $\langle a_{N-1}, b_{N-1}\rangle *_{j(N)} \langle a_N, b_N\rangle$, as $j(N)$ varies, form the derived sequence (Section 5). Continuing leftward to $\langle a_1, b_1\rangle$, we obtain the second derived sequence, the third, etc., ending with the $(N - 1)$st. In particular, the set of pairs H (regarded as unordered pairs) constitute the links of a sequence joining c and d. If $c \neq d$, at least one of these links must be nontrivial. (iii) Say

$$\langle r, s\rangle = \langle a_1, b_1\rangle *_{j(2)}(\cdots *_{j(N)}\langle a_N, b_N\rangle)\cdots).$$

For each $i = 1, \ldots, N$, a sequence from r to s will be constructed so that each link is an N translate of $\langle a_i, b_i\rangle$. The Multisequence lemma (Lemma 5.3) then asserts that a link from each sequence can be chosen so that the designated links are N-bounded; hence the $\{a_i, b_i\}$ are $(2N)$-bounded. The desired sequence from r to s for each i is simply r, e_i, s, where e_i is the common value of both coordinates in the pair $\langle e_i, e_i\rangle$ obtained by substituting a_i for b_i in the relevant position of the defining expression for $\langle r, s\rangle$. In other words, if x is a_i instead of b_i, then the pair $\langle a_1, b_1\rangle *_{j(2)} (\cdots *_{j(i)} \langle a_i, x\rangle *_{j(i+1)} (\cdots *_{j(N)} \langle a_N, b_N\rangle) \cdots) \cdots)$ becomes $\langle e_i, e_i\rangle$ instead of $\langle r, s\rangle$. Write this "variable pair" as $\langle \varphi(x), \psi(x)\rangle$. By putting the abbreviated $*$-notation back in terms of the polynomials t_j, it can be seen that φ and ψ are $(i - 1)$-translations, so are N-translations.

Since φ takes $\{a_i, b_i\}$ to $\{e_i, r\}$ and ψ takes $\{a_i, b_i\}$ to $\{e_i, s\}$, the sequence r, e_i, s has the desired property.

(The proof shows that $2N$ could have been replaced by $2N - 1$ in (c), if desired.)

7. The Application of the Criterion to Cardinality Restrictions

The main goal of this paper is to prove constructively that in a congruence-distributive equational class $\mathbf{E}$ of finite type any member with some finite number m of elements has a finite equational basis (Theorem 1.3). In Section 2, this question was reduced to the problem of finding a finite equational basis, relative to $\mathbf{E}$, for the subclass $\mathbf{E}_m$ consisting of all algebras in $\mathbf{E}$ with at most m elements. $\mathbf{E}_m$ is determined, relative to $\mathbf{E}$, by the UDE $\nu_m = (\forall x_1, ..., x_{m+1}) \mathbf{OR}_{i<j}\, x_i = x_j$. In Section 4, a criterion (Criterion 4.5) was formulated for such UDE-defined subclasses to have finite relative equational bases. The criterion, applied to ν_m in the form $(3_{\mathrm{SI}}) \Rightarrow (2)$, requires the existence of an integer k'' such that any SI algebra $A \in \mathbf{E}$ in which ν_m fails has a k''-bounded set of failure pairs for ν_m. Thus, the following lemma, which states a specific value of k'', is exactly what is needed to prove Theorem 2.1 and thereby complete the chain of reasoning.

7.1 Lemma. *Let $\mathbf{E}$ be a congruence–distributive equational class, and let m be any positive integer. Then every* SI *algebra $A \in \mathbf{E}$ with at least $m + 1$ elements contains $m + 1$ elements $c_1, ..., c_{m+1}$ such that the $m(m + 1)/2$ pairs $\{c_i, c_j\}$ $(i < j)$ are 2^{6m}-bounded in $A^{(2)}$.*

7.2 Corollary. *Let $\mathbf{E}$ be a congruence–distributive equational class of finite type, with designated Jónsson-polynomial expressions $t_0, ..., t_n$, and let B be an algebra in $\mathbf{E}$ of some finite cardinality m. Then $\mathbf{B}^e$ is determined by the union of the following finite sets of identities.*

(a) *Jónsson's identities on $t_0, ..., t_n$* (see Theorem 2.5);

(b) *a basis for the m-variable identities of B, as read off from the operation tables of the free algebra on m generators with the identities of B*;

(c) *$\mathbf{I}_k[\nu_m]$ for $k = 2^{6m}$.*

(The proof is an amalgam of Theorem 2.2, Proposition 2.4, and the application of Criterion 4.5 to ν_m.)

7.3 *Proof of Lemma* 7.1 (*first installment*). In an SI algebra $A \in \mathbf{E}$, *any* finite list of nontrivial pairs is k-bounded for *some* k, by Corollary 3.3. For the pairs $\{c_i, c_j\}$ coming from $m + 1$ distinct elements $c_1, \ldots, c_{m+1}$ in A, our task, then, is to find a value of k that can be prescribed in advance and does not depend on the choice of A. (Fortunately, within a given A the choice of $c_1, \ldots, c_{m+1}$ is ours, if we wish.) For a given SI algebra A, there are two cases, an easy one and a harder one:

Case 1. Any *two* nontrivial pairs in A are l-bounded for some known, small integer l.

Case 2. Some two nontrivial pairs in A are not l-bounded for this l, even though they are k-bounded for some k.

Of course, l remains to be specified. The proof proceeds by playing the two cases against one another: If l is carefully chosen, in Case 1 *any* distinct $c_1, \ldots, c_{m+1} \in A$ will yield a 2^{6m}-bounded set of pairs $\{c_i, c_j\}$. In Case 2, the existence of "long" k-translations (i.e., $k > l$) that cannot be shortened will force the existence of suitable $c_1, \ldots, c_{m+1}$ among the elements involved in such long k-translations.

The proper choice of l is not immediately obvious; in fact, l appears only as one entry in a list of parameters occurring naturally in the proof, each parameter being defined in terms of the preceding parameters in the list. The integer 2^{6m} is the result of a retrospective calculation at the end of this parameter list.

For future reference, here is the list of parameters and their definitions, in admittedly unmotivated form:

7.4 *Summary of Parameters*. (i) m: given.

(ii) $M = C(2m, m) - 1$, where $C(n, k)$ denotes the binomial coefficient.

(iii) $d = 2C(M + 1, 2) + 1$.

(iv) $l = dM$.

(v) $g = l \cdot \lceil \log_2 C(m + 1, 2)$, where, for a real number r, $\lceil r$ denotes the least integer $\geqslant r$.

(vi) Fact: $2^{6m} \geqslant g$.

7.5 *Details of Case* 1. Let $c_1, \ldots, c_{m+1}$ be *any* $m + 1$ distinct elements of A, and consider the $C(m + 1, 2)$ pairs $\{c_i, c_j\}$, $i < j$. By hypothesis, every two such pairs have an l-bound, further, every two such bounds have an l-bound, so that every four of the original pairs have a $2l$-bound.

This doubling process must be continued $\lceil \log_2 N$ times, where N is the number of pairs, namely $C(m+1, 2)$. (Of course, because $C(m+1, 2)$ is not a power of 2, the last time does not represent a full doubling.) Therefore, the original set of pairs is g-bounded, where $g = l \cdot \lceil \log_2 C(m+1, 2)$.

It remains to check that g can be replaced by the more convenient expression 2^{6m}, as asserted in 7.4 (vi): Stirling's formula asserts the asymptotic relation $m! \sim S(m)$, where $S(m) = (m/e)^m(2\pi m)^{1/2}$; it follows that $C(2m, m) \sim S(2m)/S(m)^2 = 2^{2m}/(\pi m)^{1/2}$, $l \sim M^3 \sim 2^{6m}/(\pi m)^{3/2}$. Since $\lceil \log_2 C(m+1, 2) \sim 2 \log_2 m$, we get $g \sim c2^{6m} \cdot (\log_2 m)/m^{3/2}$, where $c = 2/\pi^{3/2}$. To get a bound instead of an asymptotic relation, a more specific version of Stirling's formula is useful: $S(m) \leqslant m! \leqslant \rho_m S(m)$, where $\rho_m = 1/(1 - 1/(12m))$. It follows that $C(2m, m) \leqslant \rho_{2m} S(2m)/S(m)^2$. From this starting point, a bound can easily be computed for the ratio in each asymptotic relationship. The net result is a relationship $g \leqslant c' 2^{6m}(\log_2 m)/m^{3/2}$, where c' is a fraction $< \frac{1}{2}$. Because the bound on g is more of theoretical than practical value, it seems harmless to substitute the weaker but simpler bound $g \leqslant 2^{6m}$.

Case 2 will be treated in the next section.

8. The Ramsey Argument

This section continues the proof of the preceding section.

8.1 *Details of Case* 2. For l as defined in 7.4 and for a given SI algebra A, we are assuming that some two members of $A^{(2)}$ have no common l-bound in $A^{(2)}$. We wish to find $c_1, \ldots, c_{m+1} \in A$ so that the $m(m+1)/2$ pairs $\{c_i, c_j\}$ $(i < j)$ *are* g-bounded, for g as in 7.4.

The construction proceeds in seven steps.

Step 1. Choose pairs $\{a_0, b_0\}$ and $\{a_0', b_0'\}$ in $A^{(2)}$ that do have some common l-bound $\{r, s\}$ but are not k-bounded for any $k < l$.

Justification. Let $\{r_1, s_1\}$, $\{r_2, s_2\}$ be the two hypothesized pairs in $A^{(2)}$ that have *no* common l-bound in $A^{(2)}$. Since A is SI, they do have a k-bound $\{r, s\}$ for some k, which we may take to be minimal. Break each of the arrows of length k into two pieces: $\{r_1, s_1\} \to_{k-l} \{a_0, b_0\} \to_l \{r, s\}$ and $\{r_2, s_2\} \to_{k-l} \{a_0', b_0'\} \to_l \{r, s\}$. Then $\{a_0, b_0\}$, $\{a_0', b_0'\}$ are as required.

Step 2. Factor l as $l = dM$, where the values of d and M are as in 7.4 but have not yet been motivated. Name every dth intermediate pair involved in the two bounding arrows of Step 1, by writing $\{a_0, b_0\} \rightarrow_d \{a_1, b_1\} \rightarrow_d \cdots \rightarrow_d \{a_M, b_M\} = \{r, s\}$ and similarly for primed pairs. (Although these are unordered pairs, we may assume $a_M = a_M' = r$, $b_M = b_M' = s$.)

Discussion. The desired elements $\langle c_1, \ldots, c_{m+1} \rangle$ will eventually be chosen from among the a_i, or, alternatively, from among the b_i, the a_i', or the b_i'. With this goal in mind, let us consider, for each i and j with $0 \leqslant i < j \leqslant M$, the three-link sequence $\langle a_j, a_i, b_i, b_j \rangle$ of Fig. 1.

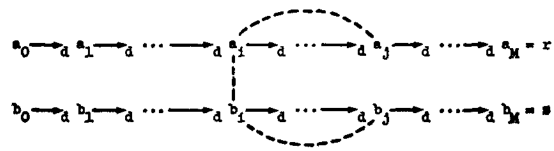

Fig. 1. Sequence of pairs for the Ramsey argument.

Although the utility of the sequence $\langle a_j, a_i, b_i, b_j \rangle$ is not yet apparent, each of its links does have an evident relevance: As i varies, the middle links $\{a_i, b_i\}$ have a common bound $\{r, s\}$; on the other hand, if all first links $\{a_j, a_i\}$ were nontrivial and had a common g-bound, then any $m + 1$ of the a_i would constitute the desired set of c_i. In fact, we would settle for a single set $\mathscr{I}$ of $m + 1$ indices such that just the pairs $\{a_j, a_i\}$ for $i, j \in \mathscr{I}$, $i < j$, have a common g-bound. Alternatively, we would accept pairs $\{b_i, b_j\}$ in place of $\{a_j, a_i\}$, or similar primed pairs.

The Multisequence lemma (Lemma 5.3) sounds potentially useful for obtaining such bounds. That lemma, however, applies only to sequences with common initial and final elements. The appropriate device to achieve such sequences is as follows.

Step 3. Recall that $\{a_j, b_j\} \rightarrow_{d(M-j)} \{r, s\}$ for each $i < j$. Let φ_{ij} be the $d(M - j)$-translation that induces this arrow. Let S_{ij} be the sequence obtained by applying φ_{ij} to the whole sequence $\langle a_j, a_i, b_i, b_j \rangle$: $S_{ij} = \langle r, \varphi_{ij} a_i, \varphi_{ij} b_i, s \rangle$. Sequences S_{ij}' are similarly defined.

Notice a somewhat suspicious circumstance appearing: φ_{ij} induces an arrow of the correct "length" to carry $\{a_j, b_j\}$ to $\{r, s\}$ but of insufficient length to carry $\{a_i, b_i\}$ to $\{r, s\}$, and yet that seems to be almost what we are attempting.

Step 4. Apply the Multisequence lemma (Lemma 5.3) to all the sequences S_{ij}, S'_{ij}, simultaneously. The number N of such sequences is $N = 2C(M + 1, 2)$. The lemma asserts the existence of a nontrivial "key link" in each sequence such that all key links have a common N-bound $\{r_0, s_0\}$ in $A^{(2)}$. If $d = N + 1 = 2C(M + 1, 2) + 1$ as in 7.4, then, the arrow from each key link to the bound $\{r_0, s_0\}$ is not quite as "long" as the arrow from each of $\{a_0, b_0\}$, $\{a_1, b_1\}$,... to the next. This defect of 1 in length will become important in the succeeding steps.

Step 5. For each $i < j$, one of the three links of S_{ij} is the key link. Examine the implication of each possibility in turn:

POSSIBILITY 1. $\{r, \varphi_{ij}a_i\} = \{\varphi_{ij}a_j, \varphi_{ij}a_i\}$ is the key link. In this case, $\{a_i, a_j\} \rightarrow_g \{r_0, s_0\}$ (a circumstance earlier held to be desirable, in the discussion after Step 2). Reasoning: By the construction of φ_{ij}, $\{a_i, a_j\} \rightarrow_{d(M-j)} \{\varphi_{ij}a_j, \varphi_{ij}a_i\} \rightarrow_{d-1} \{r_0, s_0\}$. Since $0 \leqslant i < j$ implies $j \geqslant 1, d(M - j) + (d - 1) \leqslant d(M - 1) + d = dM = l \leqslant g$. (See 7.4. Technically, we must make the harmless assumption that $m \geqslant 2$.)

POSSIBILITY 2. $\{\varphi_{ij}a_i, \varphi_{ij}b_i\}$ is the key link. In this case, one can show that $\{a_0, b_0\} \rightarrow_{l-1} \{r_0, s_0\}$ (a fact that will soon be used to represent a short-circuiting of the original condition on $\{a_0, b_0\}$). Reasoning: As mentioned after Step 3, φ_{ij} induces an arrow seemingly at least d steps too short to go from $\{a_i, b_i\}$ to the vicinity of $\{r, s\}$. Moreover, by Step 4, $\{r_0, s_0\}$ lies less than a full d-arrow past the vicinity of $\{r, s\}$. Specifically, $\{a_0, b_0\} \rightarrow_{di} \{a_i, b_i\} \rightarrow_{d(M-j)} \{\varphi_{ij}a_i, \varphi_{ij}b_i\} \rightarrow_{d-1} \{r_0, s_0\}$, and so $i < j$ implies $di + d(M - j) + (d - 1) \leqslant d(j - 1) + d(M - j) + d - 1 = DM - 1 = l - 1$.

POSSIBILITY 3. This case is the same as Possibility 1, except that $\{b_i, b_j\}$ replaces $\{a_i, a_j\}$.

Each of the sequences S'_{ij} yields three corresponding cases as well.

Of course, these three possibilities, conceivably occurring independently for so many pairs of indices $i < j$, represent a bewildering profusion of cases when considered simultaneously. The next two steps will produce out of this confusion a set $\mathscr{I}$ of $m + 1$ indices with the properties mentioned in the discussion preceding Step 3.

Step 6. Observe that either $(*)$ no middle link of any S_{ij} is a key link, or $(*')$ no middle link of any S'_{ij} is a key link. For by Possibility 2 of Step 5, the existence of key middle links both for unprimed and for primed sequences would imply that $\{r_0, s_0\}$ is an $(l-1)$-bound of both $\{a_0, b_0\}$ and $\{a_0', b_0'\}$, in contradiction to Step 1.

Without loss of generality, we may assume that the alternative $(*)$ holds. All primed elements can now be discarded and our attention confined to *un*primed elements.

Step 7. We are still faced with a gap between what we know and what we want. We want to find a set $\mathscr{I}$ of $m+1$ indices such that $\{a_i, a_j\} \to_g \{r_0, s_0\}$ for all $i < j$ in $\mathscr{I}$ (or, possibly, such that $\{b_i, b_j\} \to_g \{r_0, s_0\}$). We know merely that for each $i < j$, either $\{a_i, a_j\} \to_g \{r_0, s_0\}$ or $\{b_i, b_j\} \to_g \{r_0, s_0\}$. But this is a typical Ramsey-coloring problem: Regard the $M+1$ integers $0, 1, \ldots, M$ as vertices of a complete graph and color the edge $\{i, j\}$ *red* if the key link of S_{ij} is the initial link, *blue* if the key link is the final link. Ramsey's theorem asserts that there exists a set $\mathscr{I}$ of $n+1$ vertices such that all edges between vertices in $\mathscr{I}$ are a single color, provided only that $M+1 \geqslant \mathrm{RN}(m+1, m+1, 2)$, the relevant Ramsey number. Since $\mathrm{RN}(m+1, m+1, 2) \leqslant C(2m, m)$ ([29, p. 57]; note the misprinted $\leqslant$ in some printings), the choice $M = C(2m, m) - 1$ meets this condition. Thus the desired $\mathscr{I}$ exists.

To sum up: If $c_1, \ldots, c_{m+1}$ are simply the elements $\{a_i, i \in \mathscr{I}\}$ or $\{b_i, i \in \mathscr{I}\}$, depending on the outcome of the application of Ramsey's theorem, then in $A^{(2)}$ the pairs $\{c_i, c_j\}$ $(i < j)$ are g-bounded, and hence 2^{6m}-bounded, as required.

8.2 *Remark*. In the case of lattices, Discussion 5.4 provides a simplification, provided that all sequences used are chains. This can be accomplished by requiring $a_i < b_i$ for all i and, in Step 3, replacing the sequence $\langle a_j, a_i, b_i, b_j\rangle$ by the related chain $\langle a_j, (a_i \vee a_j) \wedge b_j, (b_i \vee a_j) \wedge b_j, b_j\rangle$. With these changes, d in 7.4 can be chosen to be simply 4, instead of $2C(M+1, 2)+1$. Thus, the parameters in the lattice case are less interdependent. This fact facilitated the discovery of the lattice version, which on inspection proved to be open to generalization.

9. Nonconstructive Shortcuts

Both Herrman [32] and Makkai [52] have developed ingenious proofs that amount to shortcuts around the calculations of Sections 7–8. Both succeed in characterizing any equational class in question by a finite set of first-order sentences, not all of which are identities. It then follows nonconstructively from Gödel's compactness theorem that any set of defining identities for the equational class can be replaced by a finite defining subset.

More specifically, Hermann considers only lattices, but develops a sufficient condition for the existence of a finite equational basis that applies not just to the equational class generated by a finite lattice, but to a more general kind of UDE-defined class subject to a uniform boundedness condition on pairs of quotients, with respect to $\rightarrow$.

Makkai, on the other hand, considers only the equational class generated by a finite algebra, but applies his method to the full congruence-distributive case. His method is, however, susceptible of generalization to an analog of Hermann's.

The following theorem summarizes the content of both approaches, while avoiding hypotheses that require technical preparation. (The proof and the specific connections with both approaches will be presented in the next sections.)

9.1 THEOREM. *Let* $\mathbf{E}_0$ *be a congruence–distributive equational class of finite type. Suppose that* $\mathbf{E}_0$ *is generated by a positive universal class* $\mathbf{K}$ *such that*

(a) $\mathbf{K}$ *is strictly elementary, and*

(b) *in* $\mathbf{K}$, *the property of being finitely subdirectly irreducible* (FSI) *is strictly elementary.*

Then $\mathbf{E}_0$ *itself is strictly elementary, i.e., is finitely based.*

9.2 *Remarks.* (1) By a "positive universal class" is meant a class defined by positive universal sentences, or equivalently, by UDE's. A test for this property is that the class be closed under the formation of ultraproducts, of subalgebras, and of epimorphic images [26, p. 275, Corollary 2].

(2) By a "strictly elementary" class (or property) is meant a class (or property) defined by some finite set of (first-order) sentences. A test is that both membership and non-membership in the class should be preserved under ultraproducts. By Gödel's compactness theorem, any defining set of sentences for such a class can be reduced to a finite defining subset. In Theorem 9.1, (a) and (b) mean strictly elementary in an absolute sense, and not merely "relative to $\mathbf{E}_0$."

(3) In (b) it would be enough to require that the FSI property be axiomatic (definable by some set of sentences), because this would imply

that being FSI is preserved under ultraproducts; the property of being non-FSI is *always* preserved under ultraproducts.

(4) In most applications, **K** is given in advance. Alternatively, though, if one starts with a finitely based congruence-distributive equational class **E** of finite type in which being FSI is a strictly elementary property, then the theorem can be applied to any equational subclass $\mathbf{E}_0$ of **E** that is generated by the models of finitely many positive universal sentences.

9.3 EXAMPLES. (E1). Let A be a finite algebra of finite type in a congruence–distributive equational class. Let $\mathbf{E}_0 = A^e$, and let $\mathbf{K} = \mathbf{HS}(\{A\})$, the class of epimorphic images of subalgebras of A. Then both **K** and the class of FSI algebras in **K** include only a finite number of isomorphism types, so are strictly elementary. Therefore **E** is finitely based, i.e., A has a finite equational basis.

(E2) Let **E** be a finitely based congruence–distributive equational class of finite type with the Principal Intersection Property (PIP), i.e., such that in any algebra of **E**, the intersection of two principal congruence relations is again principal. For example, **E** could be the class of all Heyting algebras or of all relation algebras [4, Sect. 2]. Then by [4, Theorem 2.15(a)] being FSI in **E** is strictly elementary. Theorem 9.1 can then be applied to equational subclasses of **E** as in Remarks 9.2(4). (See [4, Theorem 2.15(6)].)

(E3) [32]. Let **K** be the class of all modular lattices of length at most m; then Theorem 9.1 applies.

(E4) [32]. Let **K** be the class of all lattices (modular or not) of length at most 3; then Theorem 9.1 applies.

(E5) Let **K** be any class of lattice-ordered groups determined by a finite list of UDE's. Then it can be checked that Theorem 9.1 applies.

(E6) Let **K** be any class of lattice-ordered rings determined by a finite list of UDE's. Then it can be checked that Theorem 9.1 applies.

10. GENERIC CONSTRUCTS AND N-RADII

In preparation for the proof of Theorem 9.1, it will be helpful to have a more down-to-earth characterization of the classes to which that theorem applies. Two concepts, roughly corresponding to ideas of

Makkai's theory and of Herrmann's, will be useful: That of the "Nth generic axiom problem" of an equational class, and that of the "N-radius" of an equational class.

10.1 *Generic constructs.* What is the simplest possible UDE, other than an identity? One candidate would be ϵ_2, given by $\epsilon_2 : a_1 = b_1$ **OR** $a_2 = b_2$, a UDE with no variables, in which a_1, b_1, a_2, b_2 are nullary polynomial expressions. Of course, most familiar equational classes do not have four nullary (constant) operations, but extra nullary operations can be tacked on. Because this procedure does not alter congruence relations, the study of the UDE ϵ_2 then provides valuable information about the original algebra. In fact, suppose an equational basis Σ has been found for the class of, say, all lattices L with four added constants a_1, b_1, a_2, b_2 such that L satisfies ϵ_2. Then an equational basis can be obtained for *any* class of lattices (even without constant operations) defined by a UDE $\delta = (\forall \mathbf{x}) f_1(\mathbf{x}) = g_1(\mathbf{x})$ **OR** $f_2(\mathbf{x}) = g_2(\mathbf{x})$, simply by substitution of f_1, g_1, f_2, g_2 for a_1, b_1, a_2, b_2 in each identity of Σ: By 2.3, ϵ_2 and Σ coincide if L is SI, as then do δ and the new identities. This fact was the starting point for the development of the theory of [2], and even the theory of [4, Sect. 2]. If $\mathbf{E}$ is any equational class, then, and $N \geqslant 2$, let $\mathbf{E}[a_1, b_1, \dots, a_N, b_N]$ denote the new equational class obtained by imposing new nullary operations $a_1, b_1, \dots, a_N, b_N$ on algebras of $\mathbf{E}$ in all possible ways. Thus, for example, a 10-element algebra in $\mathbf{E}$ would yield 10^{2N} algebras in $\mathbf{E}[a_1, b_1, \dots, a_N, b_N]$. Let ϵ_N, the "Nth generic UDE" be given by $\epsilon_N : a_1 = b_1$ **OR**...**OR** $a_N = b_N$. The "Nth generic equational axiom problem" for $\mathbf{E}$ is to find an equational basis, relative to $\mathbf{E}[a_1, b_1, \dots, a_N, b_N]$, for the subclass determined by ϵ_N. If $\mathbf{E}$ is congruence–distributive, the Nth generic axiom problem can be solved as in [4, Sect. 4] or by the identities $\mathbf{I}[\delta_N]$ of Section 6 above. It will be shown below that the equational classes $\mathbf{E}_0$ to which Theorem 9.1 applies are essentially those for which the generic axiom problem has a *finite* solution (Comparison 11.2 (Q7)).

10.2 *The N-radius.* Let $\mathbf{E}$ be a congruence–distributive equational class, and let $N \geqslant 2$. In an SI or FSI algebra $A \in \mathbf{E}$, let the *radius* of a list of nontrivial pairs $\{a_1, b_1\}, \dots, \{a_N, b_N\}$ be the least integer k such that the N pairs are k-bounded, i.e., have a bound in $A^{(2)}$ via $\rightarrow_k$. (By Corollary 3.3, such a k exists.) Let the *N-radius* of A, $R_N(A)$, be the supremum (possibly ∞) of the radii of all lists of N nontrivial pairs from A. For a non-FSI algebra A, this definition can be salvaged: Let the N-radius of A, $R_N(A)$ be the supremum of all *bounded* lists of N

nontrivial pairs from A. (For the case of lattices with $N = 2$, this was the idea of Hermann [32] for defining the "weak projective radius.") Finally, for a class of algebras in $\mathbf{E}$, let the N-radius of the class be the supremum of the N-radii of its members.

10.3. *Basic Properties of the N-radius*

(P1) By the "two at a time, four at a time, eight at a time ..." reasoning of Case 1 in Section 7, it is easy to see that *if A is an* SI (*or* FSI) *algebra, then* $R_N(A) \leqslant (\lceil \log_2 N) R_2(A)$, where as before $\lceil r$, for real r, is the least integer $\geqslant r$. (The temptation to jump to the same conclusion for non-FSI algebras must be avoided; it is interesting to seek out the point at which the reasoning goes astray.)

(P2) *For given l, the statement "$R_N(A) \leqslant l$" can be reexpressed, "If any N given nontrivial pairs in A are $(l+1)$-bounded, then they are already l-bounded.*" Indeed, if the latter statement holds and $\{a_1, b_1\},\ldots, \{a_N, b_N\} \in A^{(2)}$ are k-bounded by $\{c, d\}$ for some $k > l$, write $\{a_i, b_i\} \to_{k-l-1} \{a_i', b_i'\} \to_{l+1} \{c, d\}$ for appropriate pairs $\{a_i', b_i'\}$. Since the latter pairs are $(l+1)$-bounded, they are l-bounded by some $\{c', d'\}$. Thus $\{a_i, b_i\} \to_{k-l-1} \{a_i', b_i'\} \to_l \{c', d'\}$, so that the pairs $\{a_i, b_i\}$ are $(k-1)$-bounded by $\{c', d'\}$. If $k-1$ is still $>l$, the process can be repeated to yield a $(k-2)$-bound, and so on until an l-bound is attained.

(P3) *For algebras of finite type and for given l, the statement "$R_N(A) \leqslant l$" is strictly elementary*: The reformulation given in (2) is expressible as a single first-order sentence.

For the case of lattices, Herrmann [32] found that the weak projective radius of a lattice is at most 2 more than the maximum weak projective radius of its SI subdirect factors. The analogous result in the present setting is this:

(P4) *If all* SI *subdirect factors of A have N-radius at most k, then the N-radius of A itself is at most $k + 2N$.* The reasoning: Let $\{a_1, b_1\},\ldots, \{a_N, b_N\} \in A^{(2)}$ have some bound $\{c, d\} \in A^{(2)}$. Then in some SI factor $B \cong A/\theta$, $\bar{c} \neq \bar{d}$ and so $\bar{a}_i \neq \bar{b}_i$ for all i as well (in obvious notation). If B is made into an algebra $B' \in \mathbf{E}[a_1, b_1, \ldots, a_N, b_N]$ by the addition of $\bar{a}_i, \bar{b}_i$ as nullary operations, then the pairs $\{\bar{a}_i, \bar{b}_i\}$ constitute a set of failure pairs for the generic UDE ϵ_N, and this set of failure pairs is bounded, by Corollary 3.3. By Theorem 4.4 (d), some identity in $\mathbf{I}_k[\epsilon_N]$

fails in B'. Since identities are preserved under homomorphism, that identity fails in A', the algebra obtained from A by regarding the a_i, b_i as nullary operations. But then Theorem 4.4 (e) asserts that A' contains a $(k + 2N)$-bounded set of failure pairs, which of course are just the $\{a_i, b_i\}$. To sum up: If N pairs in $A^{(2)}$ are bounded at all, they are $(k + 2N)$-bounded. Therefore $R_N(A) \leqslant k + 2N$. (Observe that this reasoning, which runs smoothly enough here, really depends on some rather technical calculations in Proof 6.4.)

(P5) *For a congruence–distributive equational class* $\mathbf{E}$ *of finite type and* $N \geqslant 2$, $R_N(\mathbf{E})$ *is finite if and only if, in* $\mathbf{E}$, *the property of being* FSI *is strictly elementary* (*relative to* $\mathbf{E}$). Reasoning: Suppose $R_N(\mathbf{E})$ is finite, say equal to k. By Corollary 3.3, an algebra $A \in \mathbf{E}$ is FSI if and only if any two pairs in $A^{(2)}$, or equivalently, any N pairs in $A^{(2)}$, have a bound in $A^{(2)}$. Since $R_N(\mathbf{E}) = k$, if such pairs have a bound at all they have a k-bound. Thus the FSI property is equivalent to the condition: Any N nontrivial pairs have a k-bound. But since $\mathbf{E}$ is of finite type, this condition can be expressed by a first-order sentence. For the opposite implication, let us prove the contrapositive: Suppose that $R_N(\mathbf{E})$ is *not* finite, i.e., that the N-radii of algebras in $\mathbf{E}$ can be arbitrary large or even infinite. By property (P4), even the SI members of $\mathbf{E}$ alone must have radii that are arbitrarily large or even infinite. In other words, for each n, $n = 0, 1, 2, \ldots$, there exists an SI (hence FSI) algebra $A_n \in \mathbf{E}$ such that some N pairs in $A_n^{(2)}$ are not n-bounded. Let A_∞ be any (free) ultraproduct of the A_n. $A_\infty^{(2)}$ contains N corresponding pairs. For arbitrary k, the N designated pairs of A_n are not k-bounded except for finitely many values of n; in A_∞, then, the corresponding N pairs are not k-bounded at all. Since k was arbitrary, the N pairs in A_∞ are not bounded, period. Then by Corollary 3.3, A_∞ is not FSI. Since the property of being FSI was not preserved under the formation of an ultraproduct, it cannot be strictly elementary.

10.4 *Remark.* Actually, the proof just concluded yields slightly more: If even the property of being SI is strictly elementary, or even axiomatic, then $R_N(\mathbf{E})$ is finite. Combined with (P5), this observation gives an intriguing conclusion: For congruence–distributive varieties of finite type, if the property of being SI is strictly elementary (or even axiomatic), then so is the property of being FSI. For example, a Heyting algebra H is SI when H has a largest element $e < 1$, a strictly elementary condition. H is FSI when 1 is join–irreducible, again a strictly elementary condition, as predicted. However, one cannot pass conversely from FSI to SI, as the example of vector lattices shows: Here being FSI is equivalent to being totally ordered, a strictly elementary property, and yet being SI is not axiomatic [4, Note 2.16(a)].

11. The Nonconstructive Proof

A proof of Theorem 9.1 will be presented, adapted from the ideas of Herrmann [32]. A comparison will then be made with the method of Makkai [52].

11.1 *Proof of Theorem* 9.1. We are given $\mathbf{E}_0$ generated by $\mathbf{K}$. Let $\mathbf{E}$ be an enveloping finitely based congruence–distributive equational class containing $\mathbf{E}$, say the class defined by Jónsson's identities on the polynomials t_i. Since $\mathbf{K}$ is a positive universal class, it is definable by UDE's. Further, since $\mathbf{K}$ is assumed by (a) to be strictly elementary, not merely relative to $\mathbf{E}_0$ but in an absolute sense, Gödel compactness ensures that $\mathbf{K}$ is definable by finitely many UDE's, say $\delta_1, \ldots, \delta_n$. Let $\mathbf{K}_i$ be the subclass of $\mathbf{E}$ determined by δ_i. Then $\mathbf{K}_1{}^e \cap \cdots \cap \mathbf{K}_n{}^e = \mathbf{K}^e$, as both sides of this equation have the same SI members by Jónsson's theorem (stated as Theorem 2.3 above), and equational classes are identifiable by their SI members.

Next, hypothesis (b) asserts that the property of being FSI in $\mathbf{K}$ is strictly elementary. Because the FSI members of $\mathbf{E}_0$ are all in $\mathbf{K}$ (again by Jónsson, Theorem 2.3 above), the property of being FSI in $\mathbf{E}_0$ is strictly elementary, even in an absolute sense. Choose $N = \max_i N_i$, where N_i is the number of disjugands in δ_i. Then by Property 10.3 (P5), $R_N(\mathbf{E}_0)$ is a finite integer, say k.

Claim. The algebras A in $\mathbf{E}_0$ are characterized by these three conditions: (i) $A \in \mathbf{E}$; (ii) $R_N(A) \leqslant k$; (iii) $A \models \mathbf{I}_k[\delta_i]$ for $i = 1, \ldots, n$. For suppose $A \in \mathbf{E}_0$; then (i) holds because $\mathbf{E}_0 \subseteq \mathbf{E}$, (ii) holds because $\mathbf{R}_N(\mathbf{E}_0) = k$, and (iii) holds because any algebra in $\mathbf{E}$ that satisfies δ_i also satisfies $I_l[\delta_i]$ for all l (Theorem 4.4 (a)). Conversely, suppose A satisfies (i), (ii), (iii). By (iii), for each i, A has no k-bounded set of failure pairs for δ_i (Theorem 4.4 (d)). By (ii) and an obvious inequality, $k \geqslant R_N(A) \geqslant R_{N_i}(A)$; by the definition of $R_{N_i}(A)$, A has no l-bounded set of failure pairs for δ_i for any l. Then by Theorem 4.4 (a, e) for each i, $A \in \mathbf{K}_i{}^e$, so that $A \in \bigcap_i \mathbf{K}_i{}^e = \mathbf{E}_0$. (The condition (i) was necessary for the other theorems quoted to apply.)

Now observe that the conditions (i), (ii), (iii) are all strictly elementary

—(i) because $\mathbf{E}$ is finitely based by choice, (ii) by Property 10.3 (P3), and (iii) because $\mathbf{I}_k[\delta_i]$ is finite (Theorem 4.4 (b)). Therefore $\mathbf{E}_0$ is a strictly elementary class and has a finite equational basis by Gödel's compactness theorem.

11.2 Comparison (with the method of Makkai). The ideas of Makkai can be adapted to produce a second proof of Theorem 9.1. Rather than relying on computation with arrows, Makkai relies directly on the special form of the identities constructed in [4]. The identities constructed in Section 6 of the present paper will serve just as well, so let us use the latter.

Aside from the reliance on identities, the basic outline of the second proof is exactly like that of the first. To make the correspondence especially clear, the use of boundedness and radii will be replaced in the second proof by the use of parallel concepts to be called "pseudoboundedness" and "pseudoradii."

Let $\mathbf{E}$ be a congruence–distributive equational class of finite type. Fix $N \geqslant 2$ and consider the generic UDE ϵ_N and the corresponding sets of identities $\mathbf{I}_k[\epsilon_N]$, $k = 0, 1, 2, \ldots$. For $A \in \mathbf{E}$ and $\{a_1, b_1\}, \ldots, \{a_N, b_N\} \in A^{(2)}$, regard the a_i, b_i as nullary operations and consider the set of identities $\mathbf{I}_k[\epsilon_N]$ for each k. If $A \models \mathbf{I}_k[\epsilon_N]$ for some k, let us say that the pairs $\{a_1, b_1\}, \ldots, \{a_N, b_N\}$ are *k-pseudobounded* in $A^{(2)}$. Correspondingly, if these pairs are k-pseudobounded for some k, call the least such k the *pseudoradius* of the list of pairs. Define the *N-pseudoradius* of A, $PR_N(A)$, to be the supremum over all such lists in A that happen to be pseudobounded. As before, the N-pseudoradius $PR_N(\mathbf{K})$ of a class $\mathbf{K}$ of algebras is the supremum of the N-pseudoradii of its members.

The properties of these concepts are very much like those of their earlier counterparts:

(Q1) [Cf. Lemma 3.2]. *In any algebra of* $\mathbf{E}$, *the congruence intersection of N given pairs is non-0 if and only if* they are *pseudobounded.* This assertion depends only on the fact that the identities we are using characterize the equational closure of the models of ϵ_N in $\mathbf{E}[a_1, b_1, \ldots, a_N, b_N]$ and the fact that the members of this equational closure are also characterized by the congruence condition $\theta(a_1, b_1) \cap \cdots \cap \theta(a_N, b_N) = 0$, [4, Theorem 1.5]. The previous version (Lemma 3.2) depended more directly on the computations of Section 5.

(Q2) [Cf. Corollary 3.3]. *$A \in \mathbf{E}$ is* FSI *if and only if every N nontrivial pairs in A are pseudobounded.* This is a direct corollary of (Q1), as before. Observe, however, that pseudoboundedness is not related to a quasi-order on $A^{(2)}$. Even so, k-pseudoboundedness still implies l-pseudoboundedness for all $l \geqslant k$.

(Q3) [Cf. Theorems 4.2 and 4.4(d)]. *If δ is a* UDE *with N disjugands and $\mathbf{K}$ is the subclass of $\mathbf{E}$ determined by δ, then $A \in \mathbf{K}^e$ if and only if no set of failure pairs for δ in A is pseudobounded. In fact, if A has a k-pseudobounded set of failure pairs for δ, then $\mathbf{I}_k[\delta]$ fails in A.* These assertions are consequences of the fact that $\mathbf{I}_k[\delta]$ is constructed from $\mathbf{I}_k[\epsilon_N]$ by a simple substitution of polynomials for the a_i, b_i.

(Q4) [Cf. 10.3 (P2)]. *For given l, the statement "$PR_N(A) \leqslant l$" can be reexpressed, "If any N given nontrivial pairs in A are $(l+1)$-pseudobounded, then they are already l-pseudobounded."* This statement is proved exactly as 10.3 (P2) was. In particular, a computation with arrows is used, so that the particular form of the identities in the sets $\mathbf{I}_k[\epsilon_N]$ is relevant. The identities of [4, Sect. 4] are of similar construction and so share the same property.

(Q5) [Cf. 10.3 (P3)]. *For algebras of finite type and for given l, the statement "$PR_N(A) \leqslant l$" is strictly elementary.* This follows from (Q4).

(Q6) [Cf. 10.3 (P4)]. *If all* SI *subdirect factors of A have N-pseudoradius at most k, then so does A itself.* Notice that this time there is no jump from k to $k + 2N$. The proof is as before, via ϵ_N, but this time is based on (Q2) and (Q3) above.

(Q7) [Cf. 10.3 (P5)]. *$PR_N(\mathbf{E})$ is finite if and only if the property of being* FSI *in $\mathbf{E}$ is strictly elementary (relative to $\mathbf{E}$).* The proof is as before, but (Q2) replaces Corollary 3.3.

The second proof of Theorem 9.1 now proceeds exactly as in Proof 11.1, except that the properties (Q—) replace their counterparts.

Conceivably, there may be a third proof similar to the second one, but starting, "Consider *any* set $\mathbf{J}$ of identities that solves the Nth generic axiom problem for $\mathbf{E}$." "k-pseudoboundedness" could be replaced by "S-pseudoboundedness" for finite subsets S of $\mathbf{J}$. The main difficulty would be in the search for substitutes for (Q4) and (Q5).

References

1. J. C. Abbott, Implicational algebras, *Bull. Math. Soc. Sci. Math. R. S. Roumanie* **11** (1967), 3–23; MR **39** No. 1312.
2. K. A. Baker, Equational axioms for classes of lattices, *Bull. Amer. Math. Soc.* **77** (1971), 97–102; MR **39** No. 5435.

3. K. A. BAKER, Equational bases for finite algebras, *Notices Amer. Math. Soc.* **19** (1972), 691–08–02, p. A–44.
4. K. A. BAKER, Primitive satisfaction and equational problems for lattices and other algebras, *Trans. Amer. Math. Soc.* **190** (1974), 125–150.
5. K. BAKER, Equational axioms for classes of Heyting algebras, *Algebra Universalis*, to appear.
6. R. BALBES AND A. HORN, Stone lattices, *Duke Math. J.* **37** (1970), 537–546; MR **43** No. 3181.
7. G. BIRKHOFF, On the structure of abstract algebras, *Proc. Cambridge Phil. Soc.* **31** (1935), 433–454.
8. G. BIRKHOFF, Subdirect unions in universal algebra, *Bull. Amer. Math. Soc.* **50** (1944), 764–768; MR **6**, 33.
9. G. BIRKHOFF, "Lattice Theory," 3rd ed., Amer. Math. Soc. colloq. Publ., Vol. 25, American Mathematical Society, Providence, R. I., 1967; MR **37** No. 2638.
10. G. BIRKHOFF AND R. S. PIERCE, Lattice-ordered rings, *Anal. Acad. Brasil. Ci.* **28** (1956), 41–69; MR **18**, 191.
11. G. BRUNS AND G. KALMBACH, Varieties of orthomodular lattices, *Canad. J. Math.* **23** (1971), 802–810; MR **44** No. 6565.
12. G. BRUNS AND G. KALMBACH, Varieties of orthomodular lattices, II, *Canad. J. Math.* **24** (1972), 328–337; MR **45** No. 3267.
13. C. C. CHEN AND G. GRÄTZER, Stone lattices I: construction theorems, *Canad. J. Math.* **21** (1969), 884–894; MR **39** No. 4065a.
14. C. C. CHEN AND G. GRÄTZER, Stone lattices II: structure theorems, *Canad. J. Math.* **21** (1969), 895–903; MR **39** No. 4065b.
15. P. M. COHN, "Universal algebra," Harper and Row, New York, 1965; MR **31** No. 224.
16. P. CRAWLEY AND R. P. DILWORTH, Algebraic Theory of Lattices, Prentice–Hall, Englewood Cliffs, N. J., 1973.
17. A. DAY, A characterization of modularity for congruence lattices of algebras, *Canad. Math. Bull.* **12** (1969), 167–173; MR **40** No. 1317.
18. A. DAY, Splitting algebras and a weak notion of projectivity, *Algebra Universalis* **5** (1975), 153–162.
19. G. EPSTEIN, The lattice theory of Post algebras, *Trans. Amer. Math. Soc.* **95** (1960), 300–317; MR **22** No. 3701.
20. T. EVANS, Identical relations in loops. I, *J. Austral. Math. Soc.* **12** (1971), 275–286; MR **45** No. 6967.
21. T. EVANS, Identities and relations in commutative Moufang loops, *J. of Algebra* **31** (1974), 508–513; MR **50** No. 523.
22. T. EVANS, Some remarks on finitely based varieties of rings, preprint.
23. A. L. FOSTER AND A. F. PIXLEY, Algebraic and equational semi-maximality; equational spectra. II, *Math. Z.* **93** (1966), 122–133.
24. E. FRIED AND G. GRÄTZER, A nonassociative extension of the class of distributive lattices, *Pacific J. Math.* **49** (1973), 59–78.
25. L. FUCHS, "Partially Ordered Algebraic Systems," Pergamon Press, New York, Addison–Wesley, Reading, Mass., 1963; MR **30** No. 2090.
26. G. GRÄTZER, "Universal Algebra," Van Nostrand, Princeton, N. J., 1968. MR **40** No. 1320.
27. G. GRÄTZER, "Lattice Theory: First Concepts and Distributive Lattices," Freeman, San Francisco, 1971.

28. P. -A. Grillet, Translations and congruences in lattices, *Acta Math. Hung.* **19** (1968), 147–162; MR **37** No. 113.
29. M. Hall, Jr.,"Combinatorial Theory," Blaisdell, Waltham, Mass., 1967; MR **37** No. 80.
30. L. Henkin and A. Tarski, Cylindric algebras, in "*Proc. Symp. in Pure Math*," Vol. 2, pp. 83–113, American Mathematical Society, Providence, R.I., 1961; MR **23** No. A1564.
31. L. Henkin, D. Monk, and A. Tarski, "Cylindric Algebras. Part I, North-Holland, Amsterdam, 1971.
32. C. Herrmann, Weak (projective) radius and finite equational bases for classes of lattices, *Algebra Universalis* **3** (1974), 51–58.
33. S. S. Holland, Jr., Current interest in orthomodular lattices, *in* Trends in lattice theory, (Sympos. U. S. Naval Academy, Annapolis, Md., 1966), pp. 41–126, J. C. Abbott, ed., Van Nostrand Reinhold, New York, 1970; MR **42** No. 7569.
34. D.-X. Hong, Covering relations among lattice varieties, *Pacific J. Math.* **40** (1972), 575–603.
35. J. R. Isbell, Notes on ordered rings, *Algebra Universalis* 1 (1971/72), 393–399; MR **45** No. 5055.
36. B. Jónsson, Algebras whose congruence lattices are distributive, *Math. Scand.* **21** (1967), 110–121; MR **38** No. 5689.
37. B. Jónsson, Topics in universal algebra, Lecture Notes in Mathematics, Vol. 250, Springer-Verlag, Berlin/New York, 1972.
38. B. Jónsson and A. Tarski, Boolean algebras with operators. II, *Amer. J. Math.* **74** (1952), 127–162; MR **13**, 524.
39. T. Katriňák, Die Kennzeichnung der distributiven pseudokomplementären Halbverbände, *J. Reine Angew. Math.* **241** (1970), 160–179; MR **41** No. 5253.
40. T. Katriňák, Die Kennzeichnung der beschränkten Brouwerschen Verbände, *Csech. Math. J.* **22** (1972), 427–434.
41. T. Katriňák, Über eine Konstruktion der distributiven pseudokomplementären Verbände, *Math. Nachr.* **53** (1972), 85–99.
42. T. Katriňák, Primitive Klassen von modularen S-Algebren, *J. Reine Angew. Math.* **261** (1973), 55–87.
43. T. Katriňák, Subdirectly irreducible modular p-algebras, *Algebra Universalis* **2** (1972), 166–173.
44. L. G. Kovács and M. F. Newman, Cross varieties of groups, *Proc. Roy. Soc. London Ser. A*, **292** (1966), 530–536; MR **33** No. 2715.
45. R. L. Kruse, Identities satisfied by a finite ring, *J. of Algebra* **26** (1973), 298–318; MR **48** No. 4025.
46. H. Lakser, Principal congruences of pseudocomplemented distributive lattices, *Proc. Amer. Math. Soc.* **37** (1973), 32–36.
47. K. B. Lee, Equational classes of distributive pseudo-complemented lattices, *Canad. J. Math.* **22** (1970), 881–891; MR **41** No. 3337.
48. R. C. Lyndon, Identities in two-valued calculi, *Trans. Amer. Math. Soc.* **71** (1951), 457–465; MR **13**, 422.
49. R. C. Lyndon, Identities in finite algebras, *Proc. Amer. Math. Soc.* **5** (1954), 8–9; MR **15**, 676.
50. S. Oates MacDonald, Various varieties, Pure Mathematics Preprint, No. 26, Dept. of Math., University of Queensland, Brisbane, Australia, 1972.

51. S. OATES MACDONALD, Laws in finite strictly simple loops, Pure Mathematics Preprint, No. 34, Dept. of Math., University of Queensland, Brisbane, Australia, 1973.
52. M. MAKKAI, A proof of Baker's finite-base theorem on equational classes generated by finite elements of congruence distributive varieties, *Algebra Universalis* 3 (1973), 174–181.
53. A. I. MAL'CEV, On the general theory of algebraic systems, *Math. Sb.* (*N.S.*) **35** (77) (1954), 3–20; English transl., *Amer. Math. Soc. Transl.* (2) **27** (1963), 125–142; MR **27** No. 1401.
54. A. I. MAL'CEV, The Metamathematics of Algebraic Systems, *In* "Studies in Logic and the Foundations of Mathematics," Vol. 66, North–Holland, Amsterdam, 1971.
55. A. I. MAL'CEV, Algebraic Systems, *In* "Die Grundlehren der math. Wissenschaften," Vol. 192, Springer–Verlag, New York, 1972; MR **44** No. 142.
56. C. G. MCKAY, On finite logics, *Nederl. Akad. Wetensch. Proc. Ser. A* **70** = *Indag. Math.* **29** (1967), 363–365; MR **35** No. 6524.
57. R. MCKENZIE, Equational bases for lattice theories, *Math. Scand.* **27** (1970), 24–38; MR **43** No. 118.
58. R. MCKENZIE, Equational bases and non-modular lattice varieties, *Trans. Amer. Math. Soc.* **174** (1972), 1–44.
59. R. MCKENZIE, Some unsolved problems between lattice theory and equational logic, preprint.
60. G. MCNULTY, The decision problem for equational bases of algebras, Thesis, University of California, Berkeley, 1972.
61. G. MICHLER AND R. WILLE, Die primitiven Klassen arithmetischer Ringe, *Math. Z.* **113** (1970), 369–372; MR **41** No. 5420.
62. A. MITSCHKE, Implication algebras are 3-permutable and 3-distributive, *Algebra Universalis* **1** (1972), 182–186.
63. D. MONK, On equational classes of algebraic versions of logic. I, *Math. Scand.* **27** (1970), 53–71; MR **43** No. 6065.
64. V. L. MURSKIĬ, The existence in three-valued logic of a closed class with finite basis not having a finite complete set of identities, *Dokl. Akad. Nauk SSSR* **163** (1965), 815–818; English transl. *Soviet Math. Dokl.* **6** (1965), 1020–1024; MR **32** No. 3998.
65. S. OATES AND M. B. POWELL, Identical relations in finite groups, *J. Algebra* **1** (1964), 11–39; MR **28** No. 5108.
66. R. PADMANABHAN AND R. W. QUACKENBUSH, Equational theory of algebras with distributive congruences, *Proc. Amer. Math. Soc.* **41** (1973), 373–377; MR **48** No. 3845.
67. P. PERKINS, Bases for equational theories of semigroups, *J. Algebra* **11** (1969), 293–314; MR **38** No. 2232.
68. R. S. PIERCE, "Introduction to the Theory of Abstract Algebras," Holt, Rinehart and Winston, New York, 1968; MR **37** No. 2655.
69. A. F. PIXLEY, Distributivity and permutability of congruence relations in equational classes of algebras, *Proc. Amer. Math. Soc.* **14** (1963), 105–109, MR **26** No. 3630.
70. A. F. PIXLEY, Completeness in arithmetical algebras, *Algebra Universalis* **2** (1972), 179–196.
71. R. W. QUACKENBUSH, Equational classes generated by finite algebras, *Algebra Universalis* **1** (1971/72), 265–266; MR **45** No. 3295.

72 H. Rasiowa and R. Sikorski, "The Mathematics of Metamathematics," 3rd ed., Monografia Matematyczne t. 41, Warszawa, 1970.
73. A. Robinson, "Introduction to Model Theory and to the Metamathematics of Algebra," 2nd ed., North-Holland, Amsterdam, 1965; MR **36** No. 3642.
74. G. Szasz, Translationen der Verbände, *Acta Fac. Rerum Nat. Univ. Comenian.* **5** (1961), 449–453; MR **24** No. A2542.
75. G. Szasz, "Introduction to Lattice Theory," Academic Press, New York, 1963; MR **29** No. 3396.
76. A. Tarski, On the calculus of relations, *J. Symbolic Logic* **6** (1941), 73–89; MR **3**, 130.
77. A. Tarski, Equational logic and equational theories of algebras, Proc. Logic Colloq., Hanover, 1966, publ. as H. A. Schmidt *et al.*, "Contributions to Mathematical Logic," North-Holland, Amsterdam, 1968, pp. 275–288; MR **38** No. 5692.
78. W. Taylor, Residually small varieties, *Algebra Universalis* **2** (1972), 33–53.
79. T. Traczyk, An equational definition of a class of Post algebras, *Bull. Acad. Polon., Sci. Sér. Sci. Math. Astronom. Phys.* **12** (1964), 147–149; MR **28** No. 5015.
80. A. S. Troelstra, On intermediate propositional logics, *Indag. Math.* **27** (1965), 141–152; MR **30** No. 4674.
81. V. V. Visin, Identity transformations in a four-valued logic (Russian), *Dokl. Acad. Nauk SSSR* **150** (1963), 719–721; English transl. *Sov. Math. Dokl.* **4** (1963), 724; MR **34** No. 1176.
82. H. Werner and R. Wille, Charakterisierung der primitiven Klassen arithmetischer Ringe, *Math. Z.* **115** (1970), 197–200; MR **42** No. 318.
83. R. Wille, Primitive Länge und primitive Weite bei modularen Verbänden, *Math. Z.* **108** (1969), 129–136; MR **39** No. 2672.
84. R. Wille, Variety invariants for modular lattices, *Canad. J. Math.* **21** (1969), 279–283; MR **39** No. 2671.
85. R. Wille, Kongruenzklassengeometrien, Lecture Notes in Mathematics, Vol. 113, Springer-Verlag, Berlin/New York, 1970; MR **41** No. 6759.
86. R. Wille, Primitive subsets of lattices, *Algebra Universalis* **2** (1972), 95–98.